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Foundations of the Genetics of Elastoplastic Materials

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Abstract: This paper develops a variational-spectral framework for describing stress–strain curves of elastoplastic materials. The proposed terminology is introduced not as an a priori metaphor, but as a compact language for mathematical objects that have already been constructed within the model. The starting point is a variational formulation leading to an Euler-Lagrange equation, an associated spectrum of admissible boundary conditions, and the corresponding solution space. We show that this differential equation already contains the principal information about the material: the order of the model, the set of governing parameters, the characteristic roots, and the resulting basis functions of the solution space. On this basis, the paper consistently introduces the function space of solutions, its basis elements, orthonormal rearrangements of that basis, and the coefficients in the stress expansion. Special attention is given to continuous phenotype adaptation, operator-governed adaptation, spiral structures in the internal coefficient space, the genetic interpretation of the model, and prospects for AI-based structural recognition of materials.

Keywords: Genetics of materials, stress-strain curve, Euler-Lagrange equation, phenotype adaptation, structural recognition.

1. Introduction

Classical constitutive descriptions of elastoplastic behavior are usually constructed either as empirical relations fitted to experimental data or as engineering approximations convenient for practical calculations [1-3]. Yet empirical accuracy alone does not resolve the central methodological problem. One must not only reproduce the stress-strain curve, but also identify the structural features of the material that determine its shape. For this reason, the present study focuses not merely on curve fitting, but on constructing a mathematical description in which the observed response appears as a consequence of the internal structure of the model.

In the proposed approach, the starting point is a variational formulation of elastoplastic response. First, a variational model is specified; next, the corresponding Euler-Lagrange equation is derived; finally, the admissible boundary conditions and the associated function space of solutions are analyzed. This order is essential. It allows one to move from an empirical description of the curve to the identification of the functional structure that already encodes the material type at the level of the mathematical model itself.

The key thesis of the paper is that the material is already encoded in the differential Euler-Lagrange equation. The first part of this encoded information is the order of the equation, which determines the dimension of the solution space. The second part is the set of parameters entering the equation, which determines its characteristic roots and, consequently, the specific form of the basis functions spanning that space. Depending on the values of these parameters, the structure of the solution may involve distinct real roots, multiple roots, or complex-conjugate roots, each leading to a different basis architecture. Thus, the Euler-Lagrange equation already fixes both the dimension of the model space and the qualitative composition of its basis elements.

Once the Euler-Lagrange equation and the spectrum of admissible boundary conditions have been specified, a natural problem arises: to represent the response as an expansion in the basis functions of the corresponding function space. At this stage, one can pass from the equation itself to a structural description of the material response. In other words, the mathematical object —

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the solution space — is constructed first, and only afterward is a given stress-strain curve interpreted as a particular realization within that space.

Only after this step does the paper introduce its special interdisciplinary terminology. These terms do not precede the mathematical construction and do not replace it. Rather, they serve as a compact descriptive language for objects that have already been defined in a rigorous mathematical way. The solution space, its original basis functions, the orthonormal rearrangements of that space, and the coefficients in the stress expansion are introduced first as strictly mathematical entities and only then receive a special interpretation. In this sense, the terminology is not a free metaphor, but a secondary language built on top of the variational-spectral construction.

An important consequence of this viewpoint is the possibility of distinguishing strictly between two types of model complication. The first is associated with changing the function space itself, that is, with changing the governing equation, its order, its parameter set, and the corresponding family of basis functions. The second is associated with increasing the complexity of the coefficient realization within one and the same solution space. For a fixed spline class, the solution space remains unchanged, whereas improved descriptive accuracy is achieved by increasing the objectively necessary number of segments on which that same space is realized by different coefficient vectors. This observation later makes it possible to interpret a segmented model not merely as a technical approximation procedure, but as a structural representation of the coefficient architecture of the material.

The logic of the paper unfolds in several stages. First, the variational foundations of the classes C^0, C^1, and C^2 are established.

2. Variational Models of Materials

This section develops the variational foundation on which the subsequent structural description of stress-strain curves is built. The basic logic is as follows. First, a variational model of the material is specified. Second, the stationarity condition for the corresponding variational form is written down. Third, the Euler-Lagrange differential equation is derived. Finally, the spectrum of admissible boundary conditions and the structure of the solution space are analyzed; these provide the basis for constructing splines of different continuity classes.

A crucial point is that, in the present approach, the material is described not by a single empirical relation σ(ε), but by a linear differential equation of fixed order.

General variational scheme.

In the scheme adopted here, the basic variables are not the stress σ(ε) itself, but its derivatives combined into the invariant forms

$$x_1 = \sigma', \quad x_2 = \varepsilon\sigma'', \quad x_3 = \varepsilon^2\sigma''' \tag{2.1}$$

The reversible part of the model is given by the quadratic form

$$\Phi[\sigma] = \int_{\varepsilon_0}^{\varepsilon_1} L_A\,(\varepsilon, \sigma', \sigma'', \sigma''')\,d\varepsilon$$

where

$$L_A = \frac{1}{2}A_{11}x_1^2 + A_{12}x_1x_2 + \frac{1}{2}A_{22}x_2^2 + A_{13}x_1x_3 + A_{23}x_2x_3 + \frac{1}{2}A_{33}x_3^2$$

The irreversible part is introduced as the non-integrable variational form

$$\delta D = \tfrac{1}{2}\int_{\varepsilon_0}^{\varepsilon_1}[B_{12}(x_1\delta x_2 - x_2\delta x_1) + B_{13}(x_1\delta x_3 - x_3\delta x_1) + B_{23}(x_2\delta x_3 - x_3\delta x_2)]\,d\varepsilon \tag{2.2}$$

The total variational form is written as

$$\delta J = \delta\Phi + \delta D \tag{2.3}$$

The stationarity condition takes the form δJ = 0 for all admissible variations δσ satisfying the chosen class of boundary conditions.

After integration by parts, the variation can be written as

$$\delta J = \int_{\varepsilon_0}^{\varepsilon_1} E\,(\sigma)\,\delta\sigma\,d\varepsilon + [B(\sigma, \delta\sigma)]_{\varepsilon_0}^{\varepsilon_1} \tag{2.4}$$

This expression gives rise to two fundamental objects. The first is the Euler–Lagrange equation

$$E(\sigma) = 0 \qquad (2.5)$$

which defines the local solution space. The second is the spectrum of admissible boundary conditions obtained by annihilating the boundary term. Together, the differential equation and the compatible boundary conditions form the mathematical basis for constructing local segments and assembling them into a spline.

Class C^0: a second-order model.

In the simplest case, only the first variable $x_1 = \sigma'$ is retained. Then

$$\Phi_{C^0}[\sigma] = \int_{\varepsilon_0}^{\varepsilon_1} \frac{1}{2} A_{11}(\varepsilon)\left(\sigma'(\varepsilon)\right)^2 d\varepsilon.$$

After integration by parts, one obtains

$$\delta J_{C^0} = -\int_{\varepsilon_0}^{\varepsilon_1} (A_{11}\sigma')'\,\delta\sigma\, d\varepsilon + [A_{11}\sigma'\,\delta\sigma]_{\varepsilon_0}^{\varepsilon_1}.$$

Hence, the Euler–Lagrange equation is

$$(A_{11}\sigma')' = 0.$$

If A_{11}= const, this reduces to σ" = 0. The general solution is then $\sigma(\varepsilon) = C_0 + C_1\varepsilon$, and the solution space is two-dimensional: G_{C^0} = span{1, ε}.

Spectrum of boundary conditions.

Because the equation is of second order, a unique local solution requires two boundary conditions — one at each end of the segment. In the spline setting, this means that the values of the function are naturally prescribed at the ends of each segment, while continuity of is enforced at the nodes. This is why the class naturally corresponds to piecewise linear splines of class C^0.

Greedy segmentation, the method of elastic continuations, and the objective choice of the number of segments.

Once the class C^0 has been introduced, a natural question arises: how many segments are needed to describe a given material?

For this reason, segmentation is proposed here in the form of a greedy reproduction procedure. The model starts with the minimum number of segments, after which successive splitting is performed; at each step, the partition that yields the largest gain according to a chosen accuracy criterion is selected. In the simplest version, this criterion may be the reduction of the root mean square error

$$\mathrm{RMSE} = \sqrt{\frac{1}{N}\sum_{j=1}^{N}\left(\sigma(\varepsilon_j) - \sigma_j^{\exp}\right)^2}$$

Alternatively, or in addition, one may use criteria related to the instrumental measurement error [7-9].

The substantive meaning of greedy segmentation is that it does not alter the function space itself. Each new segment is constructed in the same space. Consequently, adding a segment does not create a new basis element; it creates one more linear combination of the elements of the same function space. This leads to a natural interpretation: the number of segments is the objectively necessary number of local coefficient realizations of one and the same function space.

The literature contains a broad family of full-range, two-stage, and multistage models for stress-strain curves of metallic materials [10-12]. For stainless and ferritic stainless steels, one also finds three-stage, generalized multistage, and full-range formulations aimed at an accurate description of the complete deformation diagram [6, 12, 13]. The algorithmic problem of optimal discretization and approximation of experimental diagrams has likewise become an independent line of research [4, 7, 8].

Mechanisms of adaptation in the class C^0.

Within the class, each segment is associated with a coefficient vector in the plane of the two basis elements. If, this vector can be written in polar form

$$\mathbf{f}^{(k)} = R_k(\cos\theta_k, \sin\theta_k)$$

$$R_k = \sqrt{\left(f_1^{(k)}\right)^2 + \left(f_2^{(k)}\right)^2}$$

Thus, each segment is characterized not only by the amplitude, but also by the angle, which specifies the orientation of the coefficient vector in the plane of the basis elements.

If one assigns to each segment a characteristic strain level $\bar{\varepsilon}_k$, for example the midpoint of the segment, then the discrete set of pairs

$$(\bar{\varepsilon}_k, \theta_k), \qquad k = 1,2, \dots, M$$

may be regarded as an initial dataset for constructing a continuous function describing the reorientation of the coefficient vector as the strain increases. This function defines the simplest continuous model of adaptation in the two-dimensional case. Hence, in the class, adaptation is localized not in a change of the function space itself, but in a change of the orientation of the coefficient vector within a fixed two-dimensional space.

Examples and illustrations.

In the simplest example of a bilinear deformation diagram, the linear elastic branch and the branch near the ultimate tensile resistance correspond to two different orientations of the coefficient vector. For a branch that is nearly horizontal, the angle of the coefficient vector is close to zero. Thus, even an elementary bilinear diagram can already be interpreted as a discrete adaptive transition between two orientations of the coefficient vector within one and the same two-dimensional function space.

Class C^1: a fourth-order model.

To obtain a thermodynamically complete variational model, one must include the full quadratic form in the two independent variables and:

$$\Phi_{C^1}[\sigma] = \int_{\varepsilon_0}^{\varepsilon_1} \left[\frac{1}{2}A_{11}(\sigma')^2 + A_{12}\sigma'(\varepsilon\sigma'') + \frac{1}{2}A_{22}(\varepsilon\sigma'')^2\right] d\varepsilon$$

The irreversible part is given by the independent channel

$$\delta D_{C^1} = \frac{1}{2}\int_{\varepsilon_0}^{\varepsilon_1} B_{12}\,[\sigma'\,\delta(\varepsilon\sigma'') - (\varepsilon\sigma'')\,\delta\sigma']d\varepsilon$$

The full stationarity condition is

$$\delta J_{C^1} = \delta\Phi_{C^1} + \delta D_{C^1} = 0$$

If one first considers the reversible part, then, after integrating by parts twice, one obtains the Euler-Lagrange equation $-\frac{d}{d\varepsilon}(A_{11}\sigma' + A_{12}\varepsilon\sigma'') +$

$\frac{d^2}{d\varepsilon^2}(A_{12}\varepsilon\sigma' + A_{22}\varepsilon^2\sigma'') = 0$ (2.6)

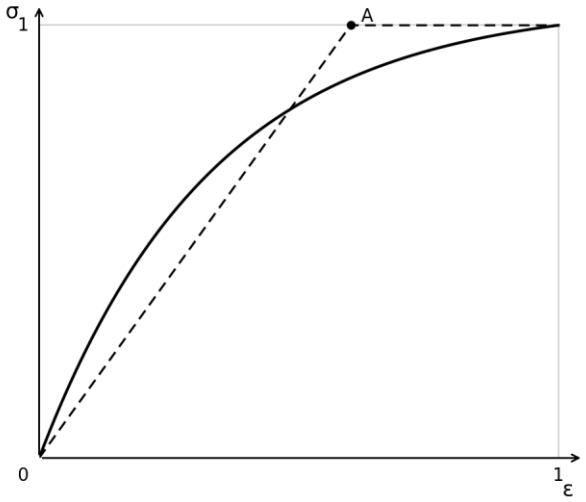

Fig. 1 A monotone curve with negative curvature and a two-segment spline.

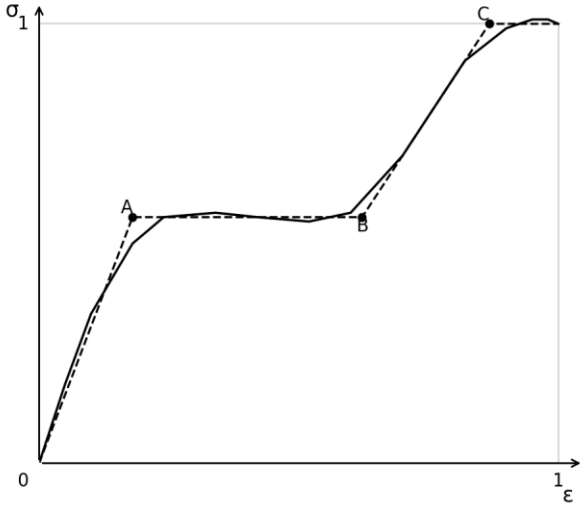

Fig. 2 A curve with a yield plateau and a four-segment spline.

For constant coefficients, this reduces to

$$A_{22}\varepsilon^2\sigma^{(4)} + 4A_{22}\varepsilon\sigma^{(3)} + (A_{12} + 2A_{22} - A_{11})\sigma'' = 0$$

Thus, one arrives at a linear fourth-order equation. In the nondegenerate case, its solution space is four-dimensional:

The main advantage of the four-dimensional basis is that it provides four expansion coefficients on each segment. This makes it possible to satisfy two boundary conditions at each end of the segment: the value of the function and the value of its first derivative at the left end, and the value of the function and the value of its first derivative at the right end. For this reason, the class

is naturally suited to the construction of a globally differentiable spline.

At the same time, the internal logic of segmentation is preserved. All segments of one and the same C^1 spline are built in the same four-dimensional function space G_{C^1}.

It should also be noted that, in this class of models, six adaptive mechanisms arise naturally, corresponding to rotations of the coefficient vector in the six pairwise coordinate planes of the function space.

Class C^2: a sixth-order variational model.

The transition to the class C^2 arises when the stress--strain curve contains a segment of positive curvature and therefore cannot be adequately described solely by continuity of the function itself and of its first derivative.

$$\sigma(\varepsilon), \qquad \sigma'(\varepsilon), \qquad \sigma''(\varepsilon)$$

This leads naturally to a sixth-order variational model. The class C^2 therefore completes the natural ladder of local models, in which each subsequent step adds not only two new degrees of freedom to the solution space, but also a new geometrically meaningful level of matching at the ends of a segment.

On an arbitrary segment, introduce the three invariant variables

$$x_1 = \sigma'(\varepsilon), \qquad x_2 = \varepsilon\,\sigma''(\varepsilon), \qquad x_3 = \varepsilon^2\,\sigma'''(\varepsilon)$$

The reversible part of the variational model is given by the full quadratic form in these three variables:

$$\Phi_{C^2}[\sigma] = \int_a^b \Big[\frac{1}{2}A_{11}x_1^2 + A_{12}x_1x_2 + \frac{1}{2}A_{22}x_2^2 + A_{13}x_1x_3 + A_{23}x_2x_3 + \frac{1}{2}A_{33}x_3^2\Big]\,d\varepsilon$$

The irreversible part is given by the three dissipation channels

$$\delta D_{C^2} = \frac{1}{2}\int_a^b [B_{12}(x_1\delta x_2 - x_2\delta x_1) + B_{13}(x_1\delta x_3 - x_3\delta x_1) + B_{23}(x_2\delta x_3 - x_3\delta x_2)]\,d\varepsilon$$

Accordingly, the full variational formulation is

$$\delta J_{C^2} = \delta\Phi_{C^2} + \delta D_{C^2} = 0$$

For subsequent integration by parts, it is convenient to introduce the notations

$$P_1 = A_{11}\sigma' + A_{12}\varepsilon\sigma'' + A_{13}\varepsilon^2\sigma''' - \frac{1}{2}B_{12}\varepsilon\sigma'' - \frac{1}{2}B_{13}\varepsilon^2\sigma'''$$

$$P_2 = A_{12}\varepsilon\sigma' + A_{22}\varepsilon^2\sigma'' + A_{23}\varepsilon^3\sigma''' + \frac{1}{2}B_{12}\varepsilon\sigma' - \frac{1}{2}B_{23}\varepsilon^3\sigma'''$$

$$P_3 = A_{13}\varepsilon^2\sigma' + A_{23}\varepsilon^3\sigma'' + A_{33}\varepsilon^4\sigma''' + \frac{1}{2}B_{13}\varepsilon^2\sigma' + \frac{1}{2}B_{23}\varepsilon^3\sigma''$$

The total variation then takes the compact form

$$\delta J_{C^2} = \int_a^b (P_1\,\delta\sigma' + P_2\,\delta\sigma'' + P_3\,\delta\sigma''')\,d\varepsilon$$

After integrating by parts three times, one obtains the variational equation on an arbitrary segment: $\delta J_{C^2} = \int_a^b E_{C^2}(\sigma)\,\delta\sigma\,d\varepsilon +$

$$+[M_0(\sigma)\,\delta\sigma + M_1(\sigma)\,\delta\sigma' + M_2(\sigma)\,\delta\sigma'']_a^b = 0$$

where

$$E_{C^2}(\sigma) = -\frac{dP_1}{d\varepsilon} + \frac{d^2P_2}{d\varepsilon^2} - \frac{d^3P_3}{d\varepsilon^3}$$

and the boundary terms determining the spectrum of admissible boundary conditions are

$$M_0(\sigma) = P_1 - \frac{dP_2}{d\varepsilon} + \frac{d^2P_3}{d\varepsilon^2},$$

$$M_1(\sigma) = P_2 - \frac{dP_3}{d\varepsilon}, \qquad M_2(\sigma) = P_3$$

Hence, at the ends of a segment, three independent boundary objects arise naturally: the coefficients at

$$\delta\sigma, \qquad \delta\sigma', \qquad \delta\sigma''$$

This is why the class allows three contact conditions to be matched at each end of the segment.

If the boundary data at the ends of the segment are prescribed as

$$\sigma(a), \sigma'(a), \sigma''(a), \quad \sigma(b), \sigma'(b), \sigma''(b)$$

then

$$\delta\sigma(a) = \delta\sigma'^{(a)} = \delta\sigma''^{(a)} = 0$$

$$\delta\sigma(b) = \delta\sigma'(b) = \delta\sigma''(b) = 0$$

and the boundary terms vanish. The stationarity

condition then yields the Euler–Lagrange equation

$$E_{C^2}(\sigma) = 0$$

The Euler–Lagrange equation.

For constant coefficients A_{ij}, B_{ij}, the Euler-Lagrange equation reduces to

$$\begin{aligned} A_{33}\varepsilon^4\sigma^{(6)} \quad & +(12A_{33}+B_{23})\varepsilon^3\sigma^{(5)} + \left(2A_{13}-A_{22}+3A_{23}+36A_{33}+\frac{15}{2}B_{23}\right)\varepsilon^2\sigma^{(4)} \\ & +(8A_{13}-4A_{22}+12A_{23}+24A_{33}-B_{12}+2B_{13}+12B_{23})\varepsilon\sigma^{(3)} \\ & +\left(A_{11}-A_{12}+6A_{13}-2A_{22}+6A_{23}-\frac{3}{2}B_{12}+3B_{13}+3B_{23}\right)\sigma'' = 0 \end{aligned}$$

This is a linear ordinary differential equation of sixth order of Euler type.

Fundamental solutions and the general solution on a segment.

Seek a solution in the power form

$$\sigma(\varepsilon) = \varepsilon^n$$

Substitution gives the characteristic equation

$$n(n-1)\,Q_4(n) = 0$$

where

$$\begin{aligned} Q_4(n) \quad & = A_{33}n^4 + (B_{23}-2A_{33})n^3 + \left(2A_{13}-A_{22}+3A_{23}-A_{33}-\frac{3}{2}B_{23}\right)n^2 \\ & +\left(-2A_{13}+A_{22}-3A_{23}+2A_{33}-B_{12}+2B_{13}+\frac{1}{2}B_{23}\right)n \\ & +\left(A_{11}-A_{12}+2A_{13}+\frac{1}{2}B_{12}-B_{13}\right) = 0 \end{aligned}$$

Two roots are universal:

$$n_0 = 0, \qquad n_1 = 1$$

while the remaining four roots are determined by

$$Q_4(n) = 0$$

In the nondegenerate case, when these four roots are pairwise distinct, the general solution on a segment has the form

$$\sigma(\varepsilon) = C_0 + C_1\varepsilon + C_2\varepsilon^{n_2} + C_3\varepsilon^{n_3} + C_4\varepsilon^{n_4} + C_5\varepsilon^{n_5}$$

If some of the roots are multiple or complex conjugate, then the corresponding fundamental solutions are replaced by the standard degenerate forms. For example, a multiple root generates logarithmic factors , while a complex-conjugate pair gives rise to the real combinations

$$\varepsilon^{\alpha}\cos(\beta\ln\varepsilon), \quad \varepsilon^{\alpha}\sin(\beta\ln\varepsilon)$$

Accordingly, the space of fundamental solutions is six-dimensional, which is precisely consistent with prescribing six independent boundary conditions at the ends of a segment.

Boundary conditions at the ends of a segment.

For a local C^2 segment, it is natural to prescribe the values of the stress, the tangent modulus, and the curvature at both ends:

$$\sigma(a) = \sigma_a, \qquad \sigma'(a) = E_a, \qquad \sigma''(a) = K_a,$$
$$\sigma(b) = \sigma_b, \qquad \sigma'(b) = E_b, \qquad \sigma''(b) = K_b$$

Substituting the general solution

$$\sigma(\varepsilon) = C_0 + C_1\varepsilon + C_2\varepsilon^{n_2} + C_3\varepsilon^{n_3} + C_4\varepsilon^{n_4} + C_5\varepsilon^{n_5}$$

into these six conditions yields a linear system for the six constants

$$C_0, C_1, C_2, C_3, C_4, C_5$$

Hence, the class C^2 is minimally sufficient for prescribing, at the ends of a segment, the three geometrically meaningful quantities

$$\sigma, \qquad \sigma', \qquad \sigma''$$

Theorem on a twice continuously differentiable spline.

Theorem.

Let the domain be partitioned by the nodes

$$a = \xi_0 < \xi_1 < \cdots < \xi_m = b$$

and let, on each segment

$$[\xi_{k-1}, \xi_k]$$

a solution $\sigma_k(\varepsilon)$ of the corresponding local sixth-order Euler--Lagrange equation be given. If, at every internal node ξ_k, the matching conditions

$$\sigma_k(\xi_k) = \sigma_{k+1}(\xi_k)$$
$$\sigma_k{}'(\xi_k) = \sigma_{k+1}{}'(\xi_k)$$
$$\sigma_k{}''(\xi_k) = \sigma_{k+1}{}''(\xi_k)$$

are satisfied, then the piecewise-defined function

$$\sigma(\varepsilon) = \sigma_k(\varepsilon), \qquad \varepsilon \in [\xi_{k-1}, \xi_k]$$

is a spline of class C^2 on the whole domain.

Proof.

On each open segment , the function is a linear combination of smooth fundamental solutions and therefore belongs to the class inside the segment. The internal nodes are the only points at which smoothness may be lost. However, owing to the matching conditions, the left and right limits of the functions themselves and of their first and second derivatives coincide at each such node:

$$\lim_{\varepsilon \to \xi_k - 0} \sigma(\varepsilon) = \lim_{\varepsilon \to \xi_k + 0} \sigma(\varepsilon)$$

$$\lim_{\varepsilon \to \xi_k - 0} \sigma'(\varepsilon) = \lim_{\varepsilon \to \xi_k + 0} \sigma'(\varepsilon)$$

$$\lim_{\varepsilon \to \xi_k - 0} \sigma''(\varepsilon) = \lim_{\varepsilon \to \xi_k + 0} \sigma''(\varepsilon)$$

Therefore, the assembled function is continuous together with its first and second derivatives throughout the domain, that is, it belongs to the class C^2. The theorem is proved.

Hierarchy of models and structural features of the material.

Thus, already at the level of the variational formulation, a natural hierarchy arises:

$$\dim G_{C^0} = 2, \qquad \dim G_{C^1} = 4, \qquad \dim G_{C^2} = 6$$

This hierarchy is determined by the order of the Euler–Lagrange equation and by the spectrum of admissible boundary conditions. A particular material within a fixed class, however, is determined by the physical constants and entering the variational model. These constants determine the coefficients of the governing equation, the form of the characteristic polynomial, and, consequently, the structure of the solution space.

Accordingly, in the proposed approach, a material is described as an object corresponding to a linear differential equation of fixed order with a definite set of physical constants. The order of the equation specifies the general class of the model, whereas the coefficients and individualize a specific material within that class.

Section 2 therefore establishes not merely a sequence of particular models, but the variational-spectral framework of the future genetics of materials. At the levels, and, three interrelated lines are already visible: the growth of the order of the equation, the growth of the dimension of the solution space, and the growth of the number of locally matchable geometric characteristics. It is precisely this triple consistency that makes it possible to pass from particular constitutive equations to a unified mathematical language in which the internal mechanisms of the material are described through the structure of the solution space and the coefficient architecture of the stress expansion.

3. Theorem on the Stepwise Growth of Smoothness Classes

The classes C^0, C^1, and C^2 considered above already show that increasing the smoothness of the model is not an external technical superstructure.

Theorem.

Consider a sequence of variational models of the classes C^0, C^1, C^2, ..., C^r, ..., constructed according to a unified principle: in the model of class C^r, the invariant variables

$$x_1 = \sigma', x_2 = \varepsilon\sigma'', x_3 = \varepsilon^2\sigma''', \dots, x_{r+1} = \varepsilon^r\sigma^{(r+1)}$$

are used; the reversible part is given by the full quadratic form in the variables; and the governing equation of the model is determined by the stationarity condition for the total variational form. Then, for the model of class C^r, the following statements hold:

- the order of the Euler-Lagrange equation is $2(r+1)$;

- the dimension of the function space of fundamental solutions is $2(r+1)$;

- in the chosen normalization, the number of independent coefficients of the normalized model is $2r$.

Consequently, under the transition the simultaneous increments

$$\Delta(\text{order}) = 2, \qquad \Delta(\dim G) = 2,$$

$$\Delta(\text{number of independent coefficients}) = 2$$

occur.

Proof.

The highest kinematic variable has the form $\varepsilon\sigma^{(r+1)}$. The full quadratic form necessarily contains the term

$$\frac{1}{2}A_{r+1,r+1}\,x_{r+1}^2 = \frac{1}{2}A_{r+1,r+1}\,\varepsilon^{2r}\left(\sigma^{(r+1)}\right)^2$$

The variation of this term contains $\delta\sigma^{(r+1)}$. To reduce the variation to a form involving $\delta\sigma$, one must integrate by parts $(r+1)$ times. Therefore, the highest quadratic term generates a derivative of order $2(r+1)$. Hence, the order of the Euler-Lagrange equation is $2(r+1)$.

A linear differential equation of order $2(r+1)$ has a local solution space of dimension $2(r+1)$. Therefore, the dimension of the solution space is $2(r+1)$.

Since the model is constructed only from derivatives of σ, rather than from the function itself, the normalized equation of class, after elimination of the common linear part, has the form

$$\sigma'' + \alpha_1\varepsilon\sigma''' + \alpha_2\varepsilon^2\sigma^{(4)} + \cdots + \alpha_{2r}\varepsilon^{2r}\sigma^{(2r+2)} = 0.$$

This shows that the number of independent coefficients of the normalized model is $2(r+1)$. All three statements are proved.

Corollary.

The theorem on stepwise growth is especially important for the inverse problem, because it determines in advance the number of independent coefficients of the equation that must be identified from the experimental dataset. Thus, the choice of model class simultaneously determines the order of the equation, the dimension of the space of fundamental solutions, and the number of unknown coefficients to be reconstructed from the data.

Remark.

For the class C^0, the formula in item 3 gives zero independent coefficients of the normalized model. This does not contradict the two-dimensionality of the solution space: in the chosen normalization, the linear part is already completely fixed, and therefore free normalized parameters first appear only from the class C^1 onward.

4. Genes, Chromosomes, Alleles, and the Function Space of Solutions

After the variational equation of the model has been constructed and its order, boundary conditions, and solution space have been analyzed, one can proceed to the special terminology that constitutes the language of the genetics of materials. Crucially, this terminology is introduced here not as an external metaphor, but as a compact interpretation of already defined mathematical objects.

Let $g_1(\varepsilon)$, $g_2(\varepsilon)$, ..., $g_m(\varepsilon)$ be a fundamental system of solutions of the corresponding differential equation of the material model. Then the solution space has the form

$$\mathcal{G} = \text{span}\{g_1, g_2, \dots, g_m\}.$$

It is precisely this space that is subsequently interpreted as the gene space of the model.

Definition.

The gene space of a material model is the function space of fundamental solutions of the corresponding differential equation.

Definition.

A gene basis is a chosen, generally non-orthonormal, basis of the gene space.

Definition.

A gene is an element of the gene basis in the function space of fundamental solutions of the governing equation of the material model.

The number of genes is equal to the dimension of the chosen basis of the function space of fundamental solutions and, consequently, to the number of roots of the characteristic equation counted with multiplicity.

This definition emphasizes that a gene is not an arbitrary analogy, but a well-defined basis function of the solution space. In particular, a simple real root gives rise to one gene, a multiple root gives rise to several linearly

independent genes, and a complex-conjugate pair gives rise, after passage to a real basis, to two real genes.

Definition.

A chromosomal basis of the gene space is any orthonormal basis of that space. The elements of such a basis are called chromosomes.

Any linearly independent set of elements of the gene space can be orthonormalized by the Gram-Schmidt procedure. Therefore, from a chosen gene basis one can always pass to one of the possible chromosomal bases

$$h_1(\varepsilon), h_2(\varepsilon), \dots, h_m(\varepsilon) \in \mathcal{G}, \qquad \langle h_i, h_j \rangle = \delta_{ij}.$$

Thus, the chromosomal basis does not create a new space and does not modify the original one; it merely reorganizes the same gene space into another, orthonormal coordinate system.

At this point, a natural clarification arises, associated with the dual way of specifying one and the same object. On the one hand, a vector may be understood in coordinate form, that is, as the set of its coefficients in a chosen basis. On the other hand, the same vector may be understood symbolically, that is, as an explicit expansion in basis functions. It is precisely this distinction that proves essential for the consistent introduction of gene and chromosomal phenotypes.

Definition.

A phenotype is the set of coefficients in the expansion of the stress in a chosen functional basis. If the basis is a gene basis, the phenotype is called a gene phenotype. If the basis is a chromosomal basis, it is called a chromosomal phenotype.

If the stress on the k-th segment is expanded in the gene basis, then it has the form

$$\sigma_k(\varepsilon) = \sum_{i=1}^{m} f_i^{(k)}\, g_i(\varepsilon) \qquad (4.1)$$

The set of coefficients

$$\mathbf{f}^{(k)} = \left(f_1^{(k)}, f_2^{(k)}, \dots, f_m^{(k)}\right)$$

is then the gene phenotype of this segment.

If the same stress is written in some chromosomal basis,

$$\sigma_k(\varepsilon) = \sum_{i=1}^{m} F_i^{(k)}\, h_i(\varepsilon),$$

then the coefficients

$$\mathbf{F}^{(k)} = \left(F_1^{(k)}, F_2^{(k)}, \dots, F_m^{(k)}\right)$$

define the chromosomal phenotype of the same segment. Hence, expansion of the stress in the gene basis and in the chromosomal basis yields different sets of coefficients, but describes one and the same phenotypic vector in one and the same gene space.

Definition.

An allele, in the extended sense, is the spectral form of realization of a single gene or of a coordinated group of genes.

This extended definition makes it possible to cover single, paired, and multiple cases in a uniform way without destroying the basic definition of the gene as an individual element of the fundamental basis.

Now let the phenotypic vector of some segment be nonzero. Then its normalized direction

$$\hat{\mathbf{c}}^{(k)} = \frac{\mathbf{c}^{(k)}}{\| \mathbf{c}^{(k)} \|}$$

may be aligned with one of the elements of some chromosomal basis of the space G.

Definition.

The canonical chromosome of a segment is an element of some chromosomal basis that coincides with the normalized phenotypic vector of that segment.

In this way, the gene space, its original gene basis, the orthonormal chromosomal basis, the coordinate representations of the phenotypic vector, and its symbolic form as an expansion in the basis are separated both terminologically and logically. This makes it possible to interpret the segmented model as a collection of phenotypic vectors of segments belonging to one and the same gene space.

Consequently, the material model contains at least two structural levels. The first level is determined by the gene space itself, that is, by the set of admissible basis mechanisms. The second level is determined by

the coefficients in the expansion of stress on the segments, that is, by the concrete phenotypes that realize these mechanisms on separate parts of the stress-strain curve. This distinction later makes it possible to separate strictly the change of solution space from the complication of coefficient architecture within a fixed space.

However, the structure of the gene space is determined not only by the choice of basis, but also by the spectral organization of its elements. The next section is devoted precisely to this issue.

5. Theorem on Factorization and Allelic Structure

In the previous section, it was established that in the present work a gene is understood as an element of the basis of the function space of fundamental solutions of the governing equation of the material model. Consequently, it is treated as an individual basis function rather than as an arbitrary spectral block.

At the same time, the structure of the characteristic equation shows that genes may naturally organize themselves into larger spectral configurations. It is precisely here that one must distinguish between two levels of description: the level of individual genes and the level of their spectral organization. The second level forms the mathematical foundation of the extended notion of the allele.

To clarify this idea, it is convenient first to consider real linear differential operators of even order with constant coefficients. In this class, the factorization theorem can be formulated in the clearest form.

Theorem.

Let

$$L_{2m} = P_{2m}(\partial)$$

be a real linear differential operator of even order 2m with constant coefficients. Then its characteristic polynomial factorizes over R into a product of m quadratic polynomials:

$$P_{2m}(\lambda) = a_{2m}Q_1(\lambda)Q_2(\lambda)\cdots Q_m(\lambda),$$

where each Q_j has real coefficients. Consequently,

$$L_{2m} = a_{2m}Q_1(\partial)Q_2(\partial)\cdots Q_m(\partial).$$

Proof.

By the fundamental theorem of algebra, the polynomial $P_{2m}(\lambda)$ factorizes over C into linear factors. Since its coefficients are real, every nonreal root appears together with its complex-conjugate counterpart. Their product yields a real quadratic factor. Real roots can also be pairwise combined into real second-degree factors, which yields the required factorization of the operator.

Remark.

This theorem does not alter the previously adopted definition of the gene. A gene remains an individual element of the functional basis of the space of fundamental solutions. The theorem merely shows that the totality of genes may possess an additional spectral organization that is invisible at the level of individual basis functions.

Corollary.

If the characteristic polynomial contains a complex-conjugate pair of roots, then the corresponding two genes naturally form a coordinated spectral pair. If a characteristic root is multiple, then the corresponding genes form a degenerate spectral block. Therefore, the allelic structure of the model is determined not by an individual gene as such, but by the form of the spectral organization of a single gene or of a coordinated group of genes.

This leads to an important distinction. A gene is an element of the functional basis of the space of fundamental solutions. An allele, in the extended sense, characterizes not an individual basis function, but the form of its spectral realization: single, paired, or multiple. Hence, the allelic structure does not replace the gene basis; rather, it is superposed on it as an additional descriptive layer.

It should be emphasized that, in the theory of materials, not only operators with constant coefficients are important, but also Euler operators. In that case, an analogous factorization logic arises after passage to the Euler operator, so that the characteristic equation again acquires a polynomial form. The theorem stated above

should therefore be regarded as a basic algebraic prototype, while the Euler case should be seen as its natural continuation within the framework of variational models of materials.

Thus, the theorem on factorization is fully consistent with the definitions adopted in Section 4: a gene remains an individual fundamental solution, whereas an allele characterizes the spectral form of realization of a single gene or of a coordinated group of genes, including the cases of complex conjugacy and multiplicity.

6. Continuous Phenotype Adaptation

The transition from a segment wise phenotype to a continuous adaptive function is the next essential step. In Subsection 2.4, the simplest adaptive scheme for the class C^0 has already been constructed: each segment was associated with a coefficient vector, and its orientation in the two-dimensional function space was specified by an angle. The discrete sequence of segment orientations therefore serves as the first prototype of a continuous adaptive function.

From discrete segment orientations to a continuous hodograph.

Let the stress on the k-th segment have the expansion given by formula (4.1); then the corresponding phenotypic vector is $\mathbf{f}^{(k)}$.

If $\| \mathbf{f}^{(k)} \|$ is nonzero, one may isolate its amplitude $R_k = \| \mathbf{f}^{(k)} \|$ and its normalized direction

$$\hat{\mathbf{c}}^{(k)} = \frac{\mathbf{f}^{(k)}}{\| \mathbf{f}^{(k)} \|}$$

It is precisely this normalized vector that determines the orientation of the segment in the internal phenotype space. In the terminology of Section-4, it coincides with the canonical chromosome of the segment. If each segment is assigned a characteristic point, for example the midpoint of the segment $\bar{\varepsilon}_k$, then one obtains the discrete set of points

$$\left(\bar{\varepsilon}_k, \hat{\mathbf{c}}^{(k)}\right), \qquad k = 1,2, \dots, M.$$

We shall call this set the discrete hodograph of segment orientations. In the class C^0, it degenerates into the set of angles $\left(\bar{\varepsilon}_k, \hat{\mathbf{c}}^{(k)}\right)$.

Continuous phenotype adaptation as interpolation of segment orientations.

The next natural step is to replace the discrete hodograph by its continuous interpolation. Instead of the finite set of orientations $\hat{\mathbf{c}}^{(k)}$, one considers a continuous function

$$\hat{\mathbf{c}} = \hat{\mathbf{c}}(\varepsilon), \qquad \| \hat{\mathbf{c}}(\varepsilon) \| = 1,$$

while the discrete amplitudes R_k are replaced by a continuous function R(ε). The stress then takes the form

$$\sigma(\varepsilon) = \sum_{i=1}^{m} f_i(\varepsilon) g_i(\varepsilon) \qquad (6.1)$$

where the functions $f_i(\varepsilon)$ are determined by the continuously varying orientation ĉ(ε) and the amplitude R(ε). Thus, continuous phenotype adaptation is the interpolatory continuation of the discrete segment wise architecture.

Theorem on continuous phenotype adaptation.

Let

$$L_n[\sigma] = \sum_{q=0}^{n} a_q(\varepsilon) \frac{d^q \sigma}{d\varepsilon^q}$$

be a linear differential operator of order n, and let $g_1(\varepsilon)$, $g_2(\varepsilon)$, ..., $g_n(\varepsilon)$ form a fundamental system of solutions of the homogeneous equation L_n[σ] = 0. Let the stress be given by formula (6.1),

$$L_n[\sigma] = S_{\text{ad}}(\varepsilon) \qquad (6.2)$$

where the adaptive source $S_{\text{ad}}(\varepsilon)$ is determined by the derivatives of the functions $f_i(\varepsilon)$.

Proof.

Substituting representation (6.1) into the operator L_n and applying Leibniz's rule leads to a sum of terms involving derivatives of $f_i(\varepsilon)$ and $g_i(\varepsilon)$.

Corollary.

If all phenotype coefficients are constant on a given segment, then the adaptive source vanishes, and the original genotypic equation holds inside that segment.

7. A Generalized Material Model With an Adaptation Operator

Continuous adaptation $S_{\mathrm{ad}}(\varepsilon)$, introduced in the previous section, describes the geometry of phenotype rearrangement, but it does not yet specify its internal law. The next step is therefore to introduce an adaptation operator, which turns the kinematic scheme into an intrinsic part of the material model.

From the continuous hodograph to the adaptation operator.

Let ĉ(ε) be the continuous orientation of the canonical chromosome of a segment, and let R(ε) be the corresponding continuous amplitude. The functions $f_i(\varepsilon)$ in the expansion (6.1) are then the coordinates of the rotating canonical chromosome in the chosen functional basis.

Definition.

An adaptation operator is a linear differential operator that determines an admissible law of change of the coordinates of a continuously rotating canonical chromosome in a chosen functional basis.

Operator-governed adaptation.

Let the basic genotypic operator have the form L_n[σ] = 0, and let its fundamental solutions form the system $g_1(\varepsilon)$, $g_2(\varepsilon)$, ..., $g_n(\varepsilon)$. Let, in addition, the continuous phenotype coefficients be determined by a linear differential adaptation operator of order m:

$$A_m[f_i] = 0, \qquad i = 1, \dots, n.$$

Then the stress retains the representation (6.1), but continuous adaptation ceases to be an arbitrary interpolation and becomes the solution of its own operator law.

Theorem on operator-governed phenotype adaptation.

Theorem.

Let form a fundamental system of solutions of the equation

$$L_n[\sigma] = 0 \tag{7.1}$$

and let form $a_1(\varepsilon)$, $a_2(\varepsilon)$, ..., $a_m(\varepsilon)$ a fundamental system of solutions of the equation

$$A_m[u] = 0 \tag{7.2}$$

Let

$$\sigma(\varepsilon) = \sum_{i=1}^{n} f_i(\varepsilon)\, g_i(\varepsilon), \quad A_m[f_i] = 0, \qquad i = 1, \dots, n$$

Then the space of all such functions σ has dimension at most nm, and therefore, locally, there exists a linear differential operator $\tilde{L}_N$ of order N ≤ nm such that

$$\tilde{L}_N[\sigma] = 0 \tag{7.3}$$

If all products $a_p(\varepsilon)\, g_i(\varepsilon)$ are linearly independent, then N = nm.

Proof.

$$f_i(\varepsilon) = \sum_{p=1}^{m} c_{ip}\ a_p(\varepsilon), \quad c_{ip} = \mathrm{const}$$

Substituting this into the representation for the stress, we obtain

$$\sigma(\varepsilon) = \sum_{i=1}^{n}\sum_{p=1}^{m} c_{ip}\ a_p(\varepsilon)\, g_i(\varepsilon)$$

Consequently, all such stress fields belong to a finite-dimensional space

$$H = \mathrm{span}\{a_p(\varepsilon) g_i(\varepsilon)\}_{p=1,\dots,m;\, i=1,\dots,n}$$

whose dimension does not exceed nm. By the general theory of linear differential equations, any finite-dimensional space of sufficiently smooth functions is locally the solution space of some linear homogeneous differential equation whose order equals its dimension. Consequently, there exists an operator $\tilde{L}_N$, where N≤nm such that relation (7.3) holds. If the products $a_p(\varepsilon)\, g_i(\varepsilon)$ are linearly independent, then N=nm.

Variational principle for adaptive functions.

Up to this point, the adaptation operator A_m has been introduced as an independent object determining an admissible law of evolution of phenotype coefficients.

Consider the simplest adaptive module, represented by a scalar function f = f(ε) on the interval [a, b]. Postulate for it a generalized variational principle in the form of the stationarity of the non-integrable variational form

$$\delta\mathcal{K}_{\mathrm{ad}}[f] = \delta\Psi_{\mathrm{ad}}[f] + \delta D_{\mathrm{ad}}[f] = 0 \tag{7.4}$$

where the reversible part is given by the functional

$$\Psi_{\mathrm{ad}}[f] = \int_a^b \left[\frac{\alpha}{2}\big(f'(\varepsilon)\big)^2 - \frac{\gamma}{2}\big(f(\varepsilon)\big)^2\right] d\varepsilon \tag{7.5}$$

and the irreversible part is given by the simplest dissipation channel

$$\delta D_{\mathrm{ad}}[f] = \frac{\beta}{2}\int_a^b (f\,\delta f' - f'\,\delta f)\,d\varepsilon \qquad (7.6)$$

Here α characterizes the "inertia" of the rearrangement of the adaptive function, γ its restoring stiffness, and β the dissipative channel of the internal rearrangement of the phenotype.

Proposition.

Let the values of the adaptive function be prescribed at the ends of the interval so that

$$\delta f(a) = \delta f(b) = 0$$

Then the stationarity condition (7.4) leads to the adaptive Euler–Lagrange equation

$$\alpha f'' + \beta f' + \gamma f = 0 \qquad (7.7)$$

Proof.

The variation of the functional (7.5) is

$$\delta\Psi_{\mathrm{ad}} = \int_a^b (\alpha f'\,\delta f' - \gamma f\,\delta f)\,d\varepsilon$$

After integration by parts, this becomes

$$\delta\Psi_{\mathrm{ad}} = [\alpha f'\,\delta f]_a^b - \int_a^b (\alpha f'' + \gamma f)\,\delta f\,d\varepsilon$$

Next, for the dissipation channel (7.6), one has

$$\delta D_{\mathrm{ad}} = \frac{\beta}{2}\int_a^b (f\,\delta f' - f'\,\delta f)\,d\varepsilon$$

Integrating the first term by parts gives

$$\delta D_{\mathrm{ad}} = \frac{\beta}{2}[f\,\delta f]_a^b - \beta\int_a^b f'\,\delta f\,d\varepsilon$$

Since δf(a) = δf(b) = 0 at the ends of the interval, all boundary terms vanish. Therefore,

$$\delta\mathcal{K}_{\mathrm{ad}} = -\int_a^b (\alpha f'' + \beta f' + \gamma f)\,\delta f\,d\varepsilon$$

By the arbitrariness of the variation δf, the equation

$$\alpha f'' + \beta f' + \gamma f = 0$$

follows. The proposition is proved.

Thus, the simplest model of the adaptive function arises as an extremal of its own generalized variational principle. Consequently, the adaptation operator may be regarded not as an external postulate, but as the result of a separate variational formulation imposed not on the stress itself, but on the phenotype coefficients.

Remark.

Under the normalization α = 1 and with the notations

$$\beta = p, \qquad \gamma = q$$

equation (7.7) takes the form

$$f'' + pf' + qf = 0 \qquad (7.8)$$

It is precisely this equation that constitutes the minimal variationally grounded model of an adaptive function, from which the oscillatory regime, the phase shift, and the spiral trajectories in the internal phenotype space naturally arise in the next section.

Remark.

For several adaptive functions, one may consider the vector

$$\mathbf{f}(\varepsilon) = \big(f_1(\varepsilon), \dots, f_r(\varepsilon)\big)^T$$

and the corresponding matrix generalization:

$$\delta\mathcal{K}_{\mathrm{ad}}[\mathbf{f}] = \delta\frac{1}{2}\int_a^b (\langle A\mathbf{f}', \mathbf{f}'\rangle - \langle C\mathbf{f}, \mathbf{f}\rangle)\,d\varepsilon + \frac{1}{2}\int_a^b (\langle B\mathbf{f}, \delta\mathbf{f}'\rangle - \langle B\mathbf{f}', \delta\mathbf{f}\rangle)\,d\varepsilon = 0$$

where A and C are symmetric matrices and B specifies the channels of internal irreversibility. In that case, one obtains the system

$$A\mathbf{f}'' + B\mathbf{f}' + C\mathbf{f} = 0$$

which serves as a natural basis for the subsequent transition to mixed adaptive channels and higher-order spiral structures.

Remark.

Sections 6 and 7 describe two successive levels of one and the same construction: first, continuous adaptation arises as an interpolation of the discrete hodograph of segment orientations; next, that interpolation acquires its own operator law. After the addition of the variational principle for adaptive functions, this second level also acquires its own variational foundation.

8. Adaptive Channels and Mixed Rotation of the Canonical Chromosome

For the class C^1, the phenotypic vector has four coordinates in the chosen functional basis. Hence, six

pairwise interaction planes are formally possible:

(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)

Once the continuous rotation of the canonical chromosome and the adaptation operator have been introduced, these six channels acquire a strict geometric interpretation.

Mixed rotation of the canonical chromosome.

Let

$$\hat{\mathbf{c}}(\varepsilon) \in \mathbb{R}^4, \qquad \| \hat{\mathbf{c}}(\varepsilon) \| = 1$$

denote the canonical chromosome of a segment, regarded as a normalized phenotypic vector in the four-dimensional coefficient space. Its continuous adaptation is then naturally described by the equation

$$\frac{d\hat{\mathbf{c}}}{d\varepsilon} = \Omega(\varepsilon)\,\hat{\mathbf{c}}, \qquad \Omega^T(\varepsilon) = -\Omega(\varepsilon) \tag{8.1}$$

where Ω(ε) is an antisymmetric operator of mixed rotation of the canonical chromosome. The antisymmetry of Ω means that the instantaneous change of the canonical chromosome occurs as an internal rotation and therefore preserves its normalization.

Six-vector of pairwise adaptive channels.

In a four-dimensional space, an antisymmetric operator has six independent components. Therefore, Ω(ε) can be written as

$$\Omega(\varepsilon) = \sum_{1\le a\le b\le 4} \omega_{ab}(\varepsilon) J_{ab} \tag{8.2}$$

where J_{ab} denotes the standard generator of rotation in the plane (a,b), and the coefficients

$$\omega_{12}(\varepsilon), \omega_{13}(\varepsilon), \omega_{14}(\varepsilon), \omega_{23}(\varepsilon), \omega_{24}(\varepsilon), \omega_{34}(\varepsilon)$$

characterize the intensity of rotation in the corresponding planes.

Thus, the six pairwise adaptive channels are naturally assembled into a single six-vector

$$\boldsymbol{\omega}(\varepsilon) = (\omega_{12}, \omega_{13}, \omega_{14}, \omega_{23}, \omega_{24}, \omega_{34})^T \tag{8.3}$$

which determines the mixed rotation of the canonical chromosome. Hence, the six adaptation channels should be understood not as six separate listed pairs, but as the coordinates of a single geometric object — an antisymmetric operator of internal rotation.

This representation is especially important because it makes explicit a deep analogy with the antisymmetric dissipation channels discussed earlier. In both cases, the algebraic structure is the same: an antisymmetric operator with six independent components lies at the foundation. The physical meaning, however, is different. In the dissipative case, the six-vector describes channels of irreversibility, whereas here it describes channels of mixed rotation of the canonical chromosome.

Orthogonalization of the mixed rotation.

Since the space of antisymmetric operators in R^4 is six-dimensional, one may introduce on it the natural scalar product

$$\langle \Omega_1, \Omega_2 \rangle = \frac{1}{2}\operatorname{tr}(\Omega_1^T \Omega_2) \tag{8.4}$$

In coordinate form, this scalar product corresponds to the standard Euclidean scalar product of six-vectors:

$$\left\langle \omega^{(1)}, \omega^{(2)} \right\rangle = \sum_{1\le a\le b\le 4} \omega_{ab}^{(1)} \omega_{ab}^{(2)} \tag{8.5}$$

Consequently, if the mixed rotation of the canonical chromosome is given by the six-vector (8.3), it can be orthogonalized by the Gram–Schmidt procedure. As a result, one obtains a new basis

$$\tilde{\Omega}^{(1)}, \tilde{\Omega}^{(2)}, \ldots, \tilde{\Omega}^{(6)}$$

in which the original mixed rotation is expanded into mutually orthogonal channels:

$$\Omega(\varepsilon) = \sum_{\alpha=1}^{6} q_\alpha(\varepsilon) \tilde{\Omega}^{(\alpha)} \tag{8.6}$$

After orthogonalization, each channel may be endowed with its own operator law, thus preparing the transition to the spiral interpretation of individual adaptive modules.

9. Spiral Structure of Adaptive Phenotypes

The previous section showed that the minimally sufficient variationally grounded model of an adaptive function is the second-order equation (7.8). The transition to spiral structure is therefore not an external geometric analogy, but a direct continuation of the previously constructed variational scheme of adaptation.

If an orthogonalized adaptive channel forms a two-dimensional primary structure and obeys a linear second-order equation in an oscillatory regime, then its internal trajectory acquires a spiral form. The second-order operator thus proves to be the minimal carrier of the primary spiral structure of the adaptive phenotype.

From an orthogonalized channel to a primary spiral structure.

After orthogonalization of the mixed rotation of the canonical chromosome, each independent channel is described by a function $q_\alpha(\varepsilon)$. To obtain a geometrically meaningful adaptive element, one should consider not an isolated scalar function, but a two-dimensional primary structure in which adaptation manifests itself as a coordinated change of two coordinates.

Theorem on the spiral character of a primary second-order structure.

Theorem.

Let two adaptive functions $f_i(\varepsilon)$ and $f_j(\varepsilon)$ form a two-dimensional primary structure and satisfy one and the same linear second-order equation with constant coefficients

$$f'' + pf' + qf = 0, \qquad p, q \in \mathbb{R},$$

and let the oscillatory regime

$$4q - p^2 > 0$$

hold. Then there exists a fundamental pair of solutions such that

$$f_i(\varepsilon) = R_0 e^{-p\varepsilon/2}\cos(\omega\varepsilon + \phi_0)$$
$$f_j(\varepsilon) = R_0 e^{-p\varepsilon/2}\sin(\omega\varepsilon + \phi_0)$$

where

$$\omega = \frac{1}{2}\sqrt{4q - p^2}$$

Consequently, the internal trajectory

$$\Gamma_{ij}(\varepsilon) = \big(f_i(\varepsilon), f_j(\varepsilon), \lambda\varepsilon\big)$$

is a spiral curve. In the special case p = 0, it becomes an exact circular helix.

Proof.

The characteristic equation is r^2 + pr + q = 0. The condition 4q - p^2 > 0 means that the roots are complex conjugates:

$$r^2 + pr + q = 0$$

Hence, the real fundamental system of solutions has the form

$$r_{1,2} = -\frac{p}{2} \pm i\omega, \qquad \omega = \frac{1}{2}\sqrt{4q - p^2}$$

Choosing two solutions shifted in phase by $\pi/2$ yields the required formulas for f_i and f_j. Substitution into Γ_{ij} then gives a spiral trajectory.

Corollary for the stress.

If the stress on a given primary structure is written as

$$\sigma_{ij}(\varepsilon) = f_i(\varepsilon)g_i(\varepsilon) + f_j(\varepsilon)g_j(\varepsilon)$$

then, in the oscillatory regime, it takes the form

$$\sigma_{ij}(\varepsilon) = R_0 e^{-p\varepsilon/2}\big[\cos(\omega\varepsilon + \phi_0)\, g_i(\varepsilon) + \sin(\omega\varepsilon + \phi_0)\, g_j(\varepsilon)\big]$$

Thus, within this structure, the stress represents an amplitude-phase rotation in the two-dimensional subspace span{g_i, g_j}.

Geometric meaning of spiral structure.

The spiral structure discussed in this section refers not to the graph of σ(ε) as a stress-strain relation, but to the internal trajectory of a primary structure in the internal phenotype space. Thus, the adaptive function may be interpreted as a continuous variation of the contributions of a pair of genes in the expansion of the stress with respect to these genes, depending on the strain level. In turn, this effect leads to its interpretation as a non-biological reflex of the material to the strain level and, more generally, to other external influences: temperature, strain rate, phase transitions, and other environmental and loading factors.

10. Double and Hierarchical Spiral Structures in the C^1 Class

In the four-dimensional phenotype space of the class C^1, adaptation is not exhausted by a single primary spiral structure. After orthogonalization of the mixed rotation of the canonical chromosome, one obtains the possibility of considering several coordinated primary spiral structures and their hierarchical superstructures.

Coordination of primary spiral structures.

Let Γ^(1)_1, Γ^(1)_2, ... denote primary spiral structures corresponding to different orthogonalized

adaptive channels. The simplest case of coordination arises when two such structures possess a common pitch parameter and a fixed phase shift.

Secondary structures as winding of primary ones.

If the axis or carrier trajectory of a primary spiral structure itself varies according to a slower law, then a secondary structure arises, naturally interpreted as the winding of a primary structure onto a slower carrier geometry.

Hierarchy of structures of order k.

If a structure of order k+1 acts as a fast internal element while its carrier trajectory itself follows a slower law, then a structure of order k arises. Hence, what appears here is not merely a collection of distinct spirals, but a successive hierarchy of internal geometric levels of adaptation.

11. Geometric Interpretation of Adaptation: The Special 3+1 Case of the Internal Space

Of special interest is the case in which the characteristic equation of the model has two real roots and one complex-conjugate pair. After passage to the real representation, the solution space remains four-dimensional, but its basis naturally splits into two parts: two genes corresponding to real roots and two real genes generated by the complex-conjugate pair.

If the complex pair has the form

$$\lambda = \alpha \pm i\beta$$

then the corresponding real solutions may be written as

$$g_3(\varepsilon) = \varepsilon^{\alpha}\cos(\beta \ln\varepsilon), \quad g_0(\varepsilon) = \varepsilon^{\alpha}\sin(\beta \ln\varepsilon)$$

Together with the two solutions corresponding to real roots, they form a real basis of the four-dimensional solution space. What is meant here is the internal space of genes and phenotypic states, not physical spacetime.

Let the continuous adaptation of the canonical chromosome be described by an antisymmetric operator Ω^T = -Ω. Then, in the basis (g_1, g_2, g_3, g_0), this operator has six independent components and may be decomposed into six generators:

$$\Omega = \omega_{12}J_{12} + \omega_{13}J_{13} + \omega_{23}J_{23} + \omega_{10}J_{10} + \omega_{20}J_{20} + \omega_{30}J_{30}$$

The first three coefficients describe rotations inside the internal three-dimensional sector, whereas the other three connect this sector to a distinguished fourth direction. Thus, the mixed rotation of the canonical chromosome admits a decomposition into spatial and temporal channels in an internal sense.

12. Nonlinear Model and Loss of Stability

If the original model is given by the nonlinear operator

$$\mathcal{N}[\sigma;\mu] = 0 \tag{12.1}$$

and the perturbed state has the form

$$\sigma = \sigma_0 + \eta u, \qquad |\eta| \ll 1,$$

then expansion in the small parameter η leads to the loss-of-stability equation

$$D\mathcal{N}_{\sigma_0}[u] = 0 \tag{12.2}$$

Thus, the linear stability equation arises as a strict linearization of the nonlinear model. In the context of the present paper, this means that the rotation of the canonical chromosome, the orthogonalized channels, and the spiral structures should be understood not as isolated geometric images, but as internal deformation regimes to be incorporated into a more general nonlinear theory.

13. Genetic Interpretation and AI Perspectives

The constructed theory naturally allows the internal architecture of material response to be interpreted as a system of coordinated basis mechanisms and coefficient states.

Genetic interpretation of the material model.

The solution space of the governing equation is interpreted as the gene space; its basis elements are interpreted as genes; the spectral forms of their organization as alleles; and the coefficients in the stress expansion as phenotypes. At the same time, the issue is not an externally biologizing language, but a structural interpretation of objects already constructed within the material model.

This interpretation makes it possible to distinguish at least three levels: the gene space, the phenotype space, and the space of adaptive phenotype transformations. The same language also makes it natural to introduce mutations and evolution. A mutation is understood as a change in the parameters of the governing equation, that is, as a displacement in the space of characteristic roots or in the space of coefficients determining the gene structure. Evolution is understood as the trajectory of such changes in the parameter space of the model. If the phenotype is continuously rearranged under loading, then the material may be regarded as a system capable of internally redistributing its response; in this extended sense, adaptation prepares the concept of material reflexivity.

AI and structural recognition of materials.

The gene space, the phenotype space, and the space of adaptive channels open up the possibility of comparing materials not only by the external similarity of their stress–strain curves, but also by their internal coefficient architecture. The proposed variational-spectral framework may therefore be regarded as a basis for future problems of structural recognition, classification, and identification of materials. It is essential that AI, in this setting, does not replace mechanical theory by statistical approximation, but operates on top of an already constructed internal structure of the material. At the same time, the simplest adaptive function is a direct analogue of sigmoidal gating in an artificial synapse [14-16]. This, in turn, motivates one to view spiral structures, understood as generalizations of adaptive functions, as prototypes of more complex AI-related neural structures [17-19].

14. Conclusion

This paper has constructed a variational-spectral framework for the genetics of elastoplastic materials, in which the Euler-Lagrange equation, its spectrum of roots, the space of fundamental solutions, and the coefficients in the stress expansion form a unified mathematical basis for introducing the notions of genes, alleles, and phenotypes. It has been shown that increasing the smoothness class of the model is accompanied by a synchronous growth in the order of the equation, the dimension of the solution space, and the number of independent coefficients of the normalized model, thereby defining a strict hierarchy of material classes. At the conceptual level, the main result may be expressed as follows: the point is not that there is an interesting terminology and several elegant constructions, but that there is a basic idea: the material is encoded in the differential equation, and the genetics of materials is the language for reading that encoded information.

The next major layer of the paper is associated with the transition from the discrete segment wise phenotype to continuous adaptation and then to operator-governed adaptation. This makes it possible to introduce the mixed rotation of the canonical chromosome, pairwise adaptive channels, their orthogonalization, primary and hierarchical spiral structures, and also a special spectral regime of the internal space of the type. The internal adaptive track culminates in the formulation of a nonlinear model and of the loss-of-stability equation as its strict linearization.

The concluding part shows that the constructed framework admits a natural genetic interpretation and opens the prospect of AI-based structural recognition of materials. The genetics of materials therefore appears not as a free metaphor, but as a mathematical language for describing the internal architecture of mechanical response; in this form, it may serve as a basis for future problems of classification, interpretation, and comparative analysis of materials.

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