

Dynamic Optimization of Portfolios 2018 to 2024

Elmo Tambosi Filho

Federal University of Santa Catarina, Florianópolis, Brazil

Investors are always willing to receive more data. This has become especially true for the application of modern portfolio theory to the institutional asset allocation process, which requires quantitative estimates of risk and return. When long-term data series are unavailable for analysis, it has become common practice to use recent data only. The danger is that these data may not be representative of future performance. Although longer data series are of poorer quality, are difficult to obtain, and may reflect various political and economic regimes, they often paint a very different picture of emerging market performance. This paper presents an application of a stochastic non-linear optimization model of portfolios including transaction costs in the Brazilian financial market. In order to have that, portfolio theory and optimal control were used as theoretical basis. The first strategy tries to allocate the whole available wealth, not considering the risk associated to portfolio (deterministic result). In this case the investor obtained profits of 7.23% a month, taking into account the three risk aversion levels during the whole planning period. On the contrary, the results from the stochastic algorithm obtain profits of 1.34% a month and 18.06% a year, if the investor has low risk aversion. The profits would be 0.88% a month and 11.02% a year for a medium risk aversion investor. And with high risk aversion, the investor obtains 0.62% a month and 7.68% a year.

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Introduction

Emerging markets have as their main feature, volatility that increases investors' potential interest for entering in such markets with the aim of possibly obtaining a higher expected return. As in Mullin (1993), for instance, just from 2000 to 2023 investments in emerging markets triplicated. This fact indicated a speculative euphoria, which occurred in other financial markets around the world.

According to Jorion and Goetzmann (1996), the last 20 years of capital market history have witnessed a dramatic expansion of opportunity for global investors, led primarily by emerging markets in Asia, South America, Africa, the Middle East, and Eastern Europe. In many countries, equity markets have grown rapidly from tiny, fledgling markets with little volume and limited international participation, to important sources of capital with short but impressive track records of share price appreciation. Although the fundamental shift in the global political environment is undoubtedly a major factor in the growth of emerging markets, consideration of a longer term view is also worthwhile.

Adding to this, Mullin (1993) suggested that share's returns in emerging markets are correlated to macroeconomic variables as well as the opening of the economy that is represented by the export growth rate.

Elmo Tambosi Filho, Dr., Federal University of Santa Catarina, Florianópolis, Brazil.

Correspondence concerning this article should be addressed to Elmo Tambosi Filho, Federal University of Santa Catarina, Florianópolis, Brazil.

The emerging countries have faced a serious liquidity crisis which causes their exclusion from international financial markets.

In this stand, Leal and Pereira e Silva (1998) showed that the issue of links and integration among the markets has important implications for the benefits of the international diversification, considering that the more markets are close economically, the more they would tend to move together.

However, they emphasized in their article that the links in international markets indicate relationship among them, which could be highlighted through the correlation coefficient, while the financial integration has a significant point that the price of assets (from the same risk class) will be the same in different markets. Thus, they concluded that two markets can be correlated without being integrated, since their movement in a group could be influenced by an external common factor.

Nowadays, it is possible to notice that the markets are more and more integrated. This fact can lead to the reduction of the international diversification benefits, considering that the crisis prominence in a certain country can be spread to another country very swiftly.

This paper presents an application considering a dynamic model of stochastic non-linear optimization of portfolios with some transaction costs for international diversification benefits. It was based on the risk-return binomial and on the quadratic utility function (*função de utilidade quadrática*—FUQ) of an investor who will determine his/her expected wealth in the end of the planning period and the risks in relation to the portfolio, also his/her risk aversion level.

The model was implemented in an Excel spreadsheet from Microsoft using optimization resources from Solver. Besides, the statistical and econometric results for 15-step forecasts, which were out of the sample, were calculated through PcGive software version 8.0. Some coefficients were estimated in order to determine the portfolio-planning period and afterwards they were implemented in the developed spreadsheet. A group of simulated mistakes were added to that based on a normal distribution. Also expected value and variance of stochastic solution were calculated and incorporated to the investor's utility function, which was optimized in a deterministic step. In such step, the proportions were also determined as well as variance and expected wealth.

Afterwards, in order to have a better understanding of the model's operation, portfolio theory and investor's utility theory were used as theoretical basis. Once the model is established, it will allow portfolio analysts to incorporate their preferences and subjective information, through a utility function, taking into consideration a group of restrictions that was incorporated to the model. Moreover, through the Excel spreadsheet, that is easily manageable; anyone can choose a portfolio that suits better his/her preferences.

This paper is divided in five sections. The second section explains the determination of the consideration coefficients for the expected wealth and the portfolio's variance through the quadratic utility function. The third section approaches briefly assets' forecasts that make part of the portfolio. The fourth section presents an empirical usage through the dynamic model of stochastic non-linear portfolio optimization with some transaction costs and finally the fifth section has some final remarks and suggestions for further studies.

The Consideration Coefficients for the Expected Wealth and the Portfolio's Variance Belonging to the Quadratic Utility Function

Brafman and Engel (2009) showed that specifying a multi-variate utility function is known to be a difficult task, and often considered a bottleneck in implementation of intelligent systems. It requires quantifying one's preferences—a non-trivial cognitive task which involves contemplating a large number of questions about the

relative desirability of uncertain outcomes, or gambles. Furthermore, the very personal and subjective nature of utility information makes it harder to reuse and learn, unlike probabilistic knowledge, which can often be learned from data and reused for various instances of a system. Yet, the preference and utility elicitation tasks must be carried out when analyzing decision problems. A number of attempts have been made to aid this elicitation process by structuring it so that either the type of questions that must be answered is simpler and/or the number of questions is smaller. Often, this process is aided by some graphical structure that captures some properties of the utility function.

The suggested dynamic model of stochastic portfolio optimization uses a quadratic utility function, in which it is supposed that individual investors make their decisions based only on the expected wealth and on the standard wealth's deviation. Therefore, considering a deterministic quadratic utility function (FUQ), we can have:

$$U = a W + b W^2 \quad (1)$$

If the mathematical average operator is used (1), we can obtain:

$$E(U) = a E(W) + b E(W^2) \quad (2)$$

As the last term of the previous expression $[E(W^2)]$ can be determined through an elementary statistical result that is $E(W^2) = \sigma^2 + E(W)^2$, then, we just have to substitute such expression from (2) in order to come up with the expression that determines the expected utility function of a portfolio, such as:

$$E(W^2) = \sigma^2 + E(W)^2 \quad (3)$$

where:

$E(U)$: Expected Utility Value;

$E(W)$: Expected Wealth Value;

σ^2 : Portfolio's Variance;

a and b : Consideration Coefficients of the Expected Wealth and Portfolio's Variance.

The consideration coefficients for expected wealth and portfolio's variance must be defined in a way to specify the area in which the subjective values of (a and b) satisfy the expression (3). Hence, the partial derivatives from the expected utility function can be taken in relation to the expected wealth and the risk, in order to calculate the marginal utilities of the function through (3) and to determine the marginal utilities' behavior. So, we can have:

$$\frac{\partial E(U)}{\partial E(W)} = a + 2bE(W) ; \text{ for: } \frac{E(U)}{E(W)} > 0 \quad (4)$$

It is possible to verify that the relationship between the expected utility and the expected wealth is positive by definition, since $a > -2bE(W)$. Also, the increase of an additional unit of expected wealth will implicate in a bigger expected utility for the investor. Besides, the partial derivatives of the expected utility in relation to the risk (marginal risk utility) will be:

$$\frac{\partial E(U)}{\partial \sigma^2} = 2b\sigma ; \text{ for: } \frac{E(U)}{\partial \sigma^2} < 0 \quad (5)$$

On the other hand, the relationship between the expected utility and the expected wealth will be negative by definition, considering an additional risk unit (the other variables are maintained constant), which will implicate in a smaller expected utility, what guarantees the established hypotheses for the model. Yet, a third

condition is necessary to preserve maximization of the analyzed function.

$$\frac{\partial^2 E(U)}{\partial E(W)^2} = 2b; \text{ for: } \frac{\partial^2 E(U)}{\partial E(W)^2} < 0 \quad (6)$$

The values assumed by the coefficient (b) in the Equation (6) (the other variables are maintained constant) will be negative, what guarantees the down concavity of the analyzed curve. It is important to mention that in all cases, the value of the coefficient (a) was maintained fixed or established. Tobin (1958) showed that the coefficient (b) can vary in the following way: $[0 < b < 1]$ if the investor is a risk lover. While if the investor is averse to the risk, it will be $[-1 < b < 0]$. However, it is also important to point out that the quadratic utility function cannot be used in the whole interval of (W), since the marginal utility of the expected wealth becomes negative, which is not appreciated by an investor.

Considering the marginal utility of the expected wealth, which was calculated through (5) and (6) and equaled to zero, it is easy to determine which coefficient values (a and b) are able to maximize the quadratic utility function (FUQ) based on the following expression:

$$b = -\frac{a}{2E(W)} \quad (7)$$

It is possible to observe through an example that: if $a = 1$ and $E(W) = 10$, the coefficient value (b) that will maximize the quadratic utility function (FUQ) will be equal to -0.05.

Forecast Model of Portfolios' Assets

A forecast model is usually built up based on temporal series analyses that are consisted of collecting previous observations of a certain item or activity in regular time intervals. Such observations can be regarding to weekly sales of a certain product, daily shares' quotations or monthly car rents, among other series.

During the analysis it is important to verify special behavior patterns, considering that such patterns will be predominant in the future, so that the forecasts can be determined for the planning period. In this paper, the estimated forecast model is established through the Ordinary Least Squares (OLS) Method including all the selected assets that are used as auxiliary tools during portfolio optimization process.

The price daily quotations of the selected assets were collected from a database called *Económica Software for Investimentos LTDA*, considering the period between 2018 and 2024.

The collected variables for the estimates are Ibovespa (an exchange market index), Savings account, Dollar, Gold and Treasury Rates (LFT) (considering just the business days of each month). All the variables were homogenized (expressed in their return rates) in order to avoid discrepancies among the characteristics of each element as references and misunderstanding in further interpretations.

In all the estimated econometric models, 22 lags were established for each variable, that is equivalent to one month (considering only the business days). It is worth to mention that there was not any problem with freedom degrees, which could affect the estimates.

Not only the binary variables for weekdays, but also the error correction method (ECM) were considered in the estimated model in order to identify long-run relationships between the saving account return variables and the treasury rates returns, which are the only first-order integrated series in the estimates.

In conclusion, models estimated through OLS will be used as basis for building up dynamic stochastic non-linear portfolios with some transaction costs.

Empirical Application of Dynamic Model of Stochastic and Non-linear Portfolio Optimization with Transaction Costs

The following subsections present the consideration coefficient for the expected wealth and the portfolio risk through the spreadsheet Excel. Also, the proposed model scenarios are highlighted.

Determination of Consideration Coefficients (*a* and *b*) of the FUQ

The problem of choosing the consideration coefficients of the expected wealth and of the portfolio variance, which will optimize determined objective function, influences strongly on the optimal path to be followed. Different considerations for the variables will result in different paths over time, due to the fact that they obtain varied criteria functions to be optimized.

The Marginal Rate of Technical Substitution (*Taxa Marginal de Substituição Técnica*—TMST) was used as measure to determine the consideration coefficients of the quadratic utility function in the optimization, as well as the investor's aversion level. It represents the ratio among the slopes of the quadratic utility function in relation to the expected wealth and to the portfolio risk. It can be seen as:

$$TMST = - \frac{\frac{\partial E(U)}{\partial E(W)}}{\frac{\partial E(U)}{\partial \sigma^2}} \quad (8)$$

The interpretation of TMST is in accordance with the level of importance considered to one of two ratios above. Through (8) we can see that if the considered level of importance for the expected wealth is higher than the risk (the other variables are maintained constant), TMST will decrease, which means that the investor possesses a low-risk aversion. Therefore, such investor should invest big part of his/her wealth on the riskiest assets, since it is possible to change an additional risk unit, as long as it is rewarded by a higher wealth. On the contrary, if a higher importance is considered for the standard deviation-risk (the other variables are maintained constant), TMST will increase, which highlights a high-risk aversion. Therefore, the investor should invest big part of his/her wealth on conservative assets.

Table 1

*Consideration Coefficients (*a* and *b*) of the Risk and of the Expected Wealth*

Aversion levels	<i>a</i>	<i>b</i>	$\frac{\partial E(U)}{\partial E(W)}$	$\frac{\partial E(U)}{\partial \sigma^2}$	<i>E(W)</i>	$\sigma_{Initial}$	TMST
Low risk aversion	1	-0.039	0.22	-0.0011	10	0.0143	-198.62
Medium risk aversion	1	-0.0105	0.79	-0.0003	10	0.0143	-2649.14
High risk aversion	1	-0.008	0.84	-0.0002	10	0.0143	-3697.05

Table 1 above shows the consideration coefficients (*a* and *b*) that will define the investor's aversion level as basis for the quadratic utility function optimization. A model with different variance-covariance matrixes in different periods of time and with some transaction costs is considered in the scenarios.

The Scenarios in the Portfolio Model

The scenarios take into account low, medium, and high individual investor's risk aversion for a stochastic no-linear dynamic portfolio with different variance-covariance matrixes in different periods of time and with some transaction costs. It is worth to emphasize that the portfolio-planning period had 15 business days of forecast.

Optimization of stochastic non-linear portfolio's model with different variance-covariance matrixes in different periods of time and with some transaction costs. In this subsection, three risk aversion levels were considered, such as: low, medium, and high investor's risk aversion, in relation to the consideration coefficients of the expected wealth's value and of the optimized portfolio's variance through the quadratic utility function (see Figures 1-3).

Scenario 1: Optimization of the stochastic model with different variance-covariance matrixes in different period of time and with some transaction costs.

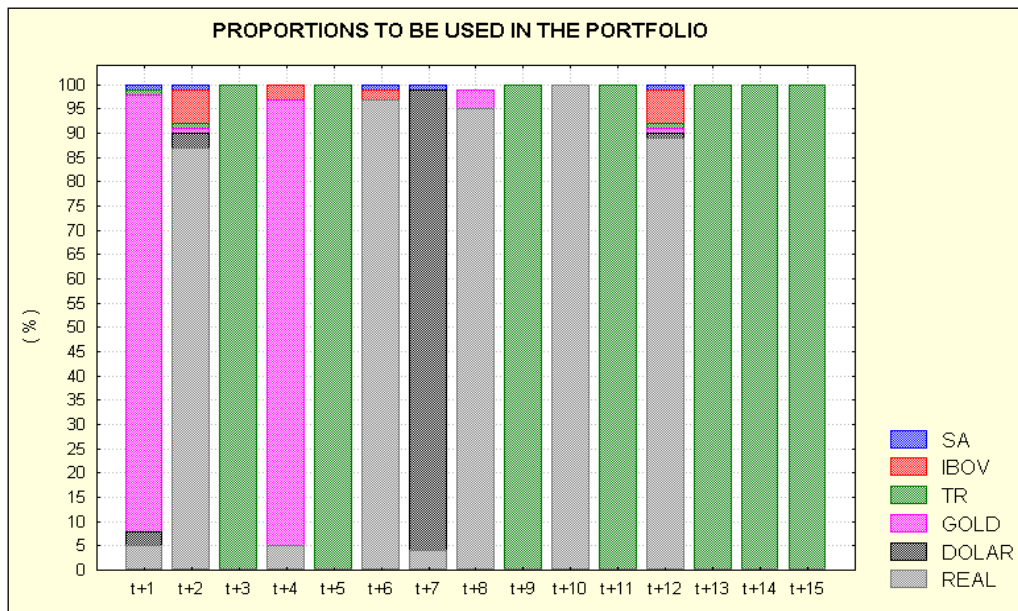


Figure 1. Optimization with low risk-aversion and with: $a = 1$ and $b = -0.039$.

Scenario 2: Optimization of stochastic model with different variance-covariance matrixes in different periods of time with some transaction costs.

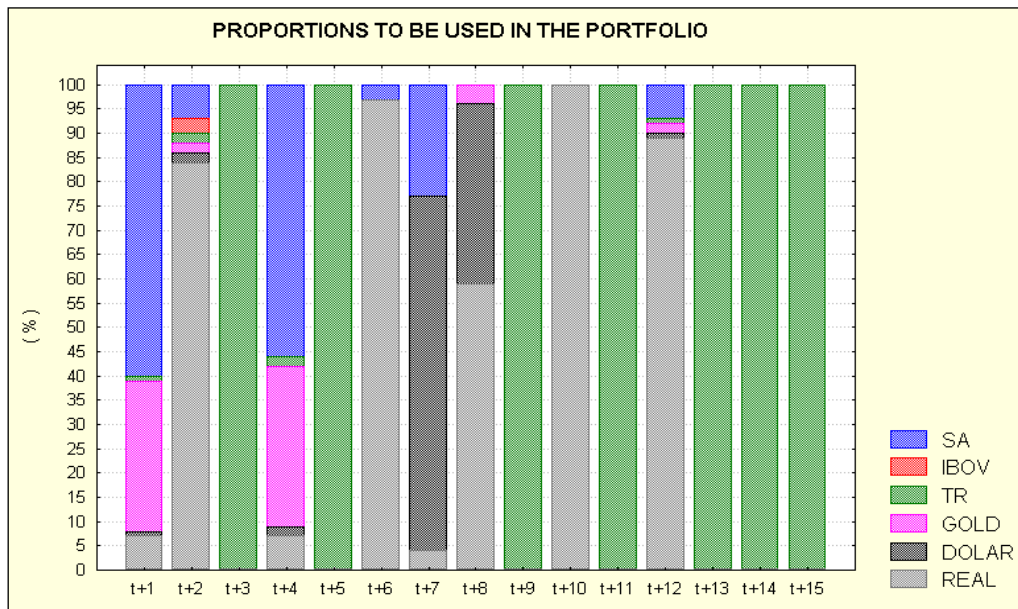


Figure 2. Optimization with medium risk-aversion and with: $a = 1$ and $b = -0.0105$.

Scenario 3: Optimization of stochastic model with different variance-covariance matrixes in different periods of time with some transaction costs.

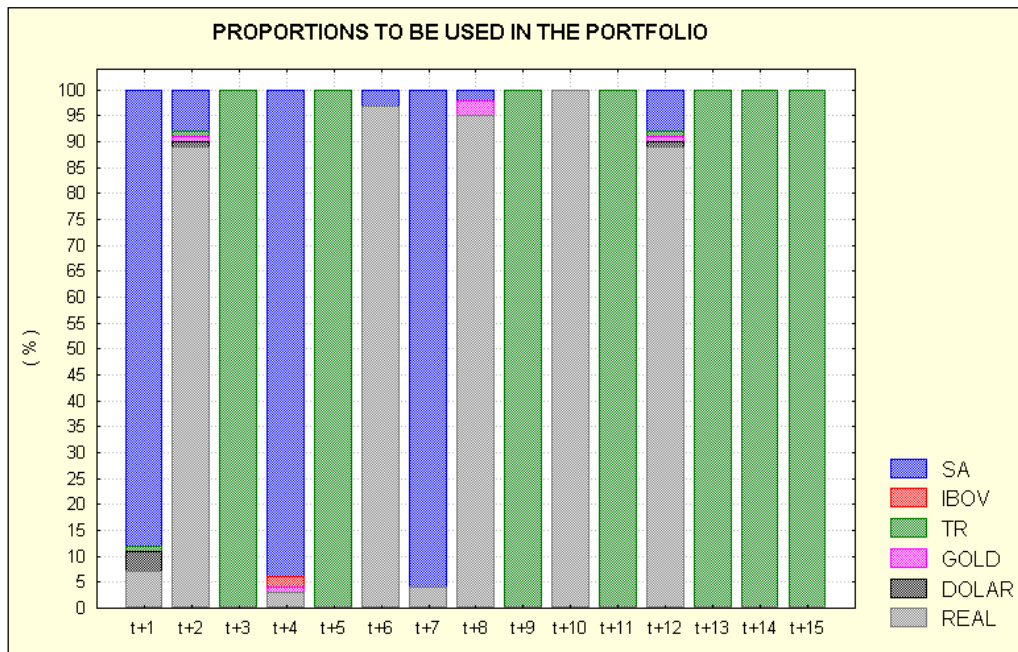


Figure 3. Optimization with high risk aversion and with: $a = 1$ and $b = -0.008$.

Table 2

Summary of the Optimized Portfolio's Scenarios with the Predicted Returns Considering: Different Variance-Covariance Matrixes in Different Periods of Time and with Some Transaction Costs

Aversion levels	a (1)	b (2)	$E(W)$ (3)	Portfolio's variance (4)	Accumulated return (%) (5)	Annual return (%) (6)	Deterministic return (%) (7)
Low aversion	1	-0.039	10.179	0.01431	1.34	18.06	7.23
Medium aversion	1	-0.0105	10.101	0.01422	0.88	11.02	7.23
High aversion	1	-0.008	10.062	0.01420	0.62	7.68	7.23

Through Table 2 it is possible to verify that: In case that the individual investor invests R\$10,00 and takes high risks (low aversion), at the end of one year, his/her portfolio will pay about 18.06%, just as it is specified in the Column (6). Yet, in case that he does not want to take any risk (high aversion), after one year, he will gain a profitability tax of about 7.68%.

Final Remarks

The purpose of this article was to accomplish a practical application through a dynamic model of stochastic and non-linear portfolio optimization with some transaction costs. It was based on the algorithm, which had been previously suggested by Hall and Stephenson (1990).

A portfolio was built with three investor's risk aversion levels. The dynamic portfolio of stochastic and non-linear optimization possesses different variance-covariance matrixes in different periods of time and with transaction costs. The algorithm that was tested in this paper showed to converge quickly in four steps at the most what made the proportions' values not modify from a step to another. Such fact caused a significant

reduction in the computational effort. In addition to that, It is worth to point out that all convergence criteria of the proposed model were covered.

This model has the assumption that each decision was associated to a normal probability distribution based on application of mathematical techniques in all simulations.

In addition to that, it was verified that in this portfolio, the three risk-free asset return rates, such as: Savings account, Treasury Rates (LFT), and Money (R\$), can be classified as dominant assets, since they increased their participation significantly in the portfolio after each optimization process. Moreover, it is possible to observe that 50% of the portfolio's assets can be classified as risk-free, since there is a great possibility that a considerable part of an investor's wealth is designated for those investments, which corroborate for the portfolio's conservatism.

Regarding to the quadratic utility function, in spite of some limitations, it is easily manageable, as long as the viability area of the consideration coefficients is defined. Besides, possible non-linearity problems, which are present in the estimated equations, should be analyzed in further studies.

In relation to transaction costs, just the CPMF (*Contribuicao Provisoria sobre Movimentacao Financeira*), which is a temporary tax over financial transactions, and Brokerage Cost were incorporated to the model. Other costs should be included in further studies.

Through analysis of returns from Table 2, we can find some investment strategies that can be designed for the investor. The first strategy tries to allocate the whole available wealth, not considering the risk associated to portfolio (deterministic result). In this case the investor obtained profits of 7.23% a month, taking into account the three risk aversion levels during the whole planning period [see Column (7)]. On the contrary, the results from of the stochastic algorithm obtained profits of 1.34% a month and 18.06% a year, if the investor had low risk aversion. The profits would be 0.88% a month and 11.02% a year for a medium risk aversion investor. And with high risk aversion, the investor obtained 0.62% a month and 7.68% a year.

Finally, it is recommended to also test other utility functions in further studies, with the aim of comparing them with the one used in this paper. In addition to that, it is recommended to include macroeconomic variables as well as to incorporate other transaction costs in the model in order to turn it even more operational.

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