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Measuring I/O-Production in Digital Economy

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According to OECD standards (United Nations, 2008), "productivity is commonly defined as a ratio of a volume measure of output to a volume measure of input use". This ratio indicates how efficiently production inputs, such as labour and capital, are being used in an economy to produce a given level of outputs. Productivity stays aside the main aggregates of national accounts, as the national income (a proxy of GDP), the total output and the circulating capital. Assume the existence of a Leontief-type national Input-Output Table with the vector of total output and the vector of intermediate inputs, situated on its right and lower border. In this paper a measure of capital productivity is proposed. It is called the productiveness, and results from the solution of a boundary value problem, elaborated for Input-Output Tables, involving the vector of total output and the vector of intermediate inputs.

Keywords: Input-Output Table (IOT), productiveness, commodity flow matrix, Perron-Frobenius theorem, Frobenius number, eigenvectors

Some National Input-Output Tables

Consider a schematic Input-Output Table (Table 1)[1] (Nathani, Schmid, & van Nieuwkoop, 2011), where the transactions z_{ij} in monetary terms are recorded at basic prices in the *supply-table* of the central part, described by the $n \times n$ *commodity flow matrix* $Z = \{z_{ij}\}$. On the right side is the use-table of final consumption, which is in reality a matrix, but here it is aggregated to a single vector $f = \{f_i\}$ of final use. Then, there is on the lower side the value added part, which is here represented by the vector of intermediate inputs $y_I = \{y_{Ij}\}$ and the vector $v = \{v_j\}$ of value added.

We work with national Input-Output Tables, the German IOT 2013, the Austrian IOT 2015, and the Swiss IOT 2014, which differ one from the other, not only with respect to the number n of sectors but also to their structure. The sectors are composed of product groups in the rows, and the same number of homogenous economic activities (branches) in the columns, classified according to the CPA system.

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[1] The notations follow the usual standards. The products or product groups are classified in the columns according to the CPA system, the economic activities are grouped in homogenous branches in the rows according to the NACE classification but are then adapted in the symmetric Input-Output Tables also to the CPA system. Germany, Austria and Switzerland operate in this way, see Nathani, Schmid, and van Nieuwkoop (2011).

Table 1

A Symmetric IOT of n Producing Sectors and a Sector of Total Final Use

Products (CPA)	Homogenous branches: buying sectors						Final use	Total output
selling sectors	S_1	S_2	...	S_j	...	S_n		
S_1	z_{11}	z_{12}	...	z_{1j}	...	z_{1n}	f_1	x_1
S_2	z_{21}	z_{22}	...	z_{2j}	...	z_{2n}	f_2	x_2
...	...	...	...	...	...	...	...	...
S_i	z_{i1}	z_{i2}	...	z_{ij}	...	z_{in}	f_i	x_i
...	...	...	...	...	...	...	...	...
S_n	z_{n1}	z_{n2}	...	z_{nj}	...	z_{nn}	f_n	x_n
Outlays (intermediate Inputs)	y_{I1}	y_{I2}	...	y_{Ij}	...	y_{In}	V = F	
value added	v_1	v_2	...	v_j	...	v_n		
Total outlays	x_1	x_2	...	x_j	...	x_n		X

One may expect that all the production sectors participate with their goods to the *means of production*. But this is not the case. For example, in the Austrian IOT 2015 there is the branch $n = 65$ and the product $n = 65$, called "Services provided by extraterritorial organisations and bodies", exhibiting exclusively zero entries, even a null-row and a null-column throughout the whole IOT! For this reason, the corresponding total output is zero, $x_{65} = 0$. The null-row and null-column of sector $n = 65$ can be skipped, because they contribute to no aggregates. Then the vector of total output, reduced by value $x_{65} = 0$, becomes positive, $x > o$, a condition we request to be fulfilled in our calculations!

The generally reducible[2] (Emmenegger et al., 2020, p. 459; Gantmacher, 1986, p. 417) semi-positive *commodity flow matrices* $Z = \{z_{ij}\} \geq O$ have therefore a maximal, real, non-negative Frobenius number $\lambda_Z \geq 0$ (Frobenius, 1912) to which are associated non-negative eigenvectors $s \geq o, Zs = \lambda_Z s$, see Gantmacher (p. 409), or, Emmenegger et al. (2020, p. 472, Theorem A.10.1). Matrix Z can be transformed with a permutation matrix P in a completely reducible "canonical form",

$$\widetilde{\mathbf{Z}} = \mathbf{P}^{-1}\mathbf{Z}\mathbf{P} = \begin{bmatrix} \widetilde{\mathbf{Z}}_{11} & \mathbf{0} & \dots & \mathbf{0} \\ \widetilde{\mathbf{Z}}_{21} & \widetilde{\mathbf{Z}}_{22} & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots \\ \widetilde{\mathbf{Z}}_{s1} & \widetilde{\mathbf{Z}}_{s2} & \dots & \widetilde{\mathbf{Z}}_{ss} \end{bmatrix}, \tag{1}$$

where $\tilde{Z}_{11}$, $\tilde{Z}_{22}$, ..., $\tilde{Z}_{SS}$ now are irreducible square matrices, not necessarily of the same order (note that the zero matrix of order 1 is irreducible).

Indeed, we need positive Frobenius numbers, and we have to strengthen the mentioned theorems. This is possible. Without giving a complete proof, the fact that there are many positive components in matrices Z, representing the substance of the economy, guarantees the positivity of the Frobenius number $\lambda_Z > 0$. The submatrices $\tilde{Z}_{ii}, i = 1, \dots s$, of matrix Z (1) are irreducible square matrices. The maximal eigenvalue of Z (Frobenius number) is equal to the maximal eigenvalue of one of the submatrices $\tilde{Z}_{ii}$, see Emmenegger et al. (2020, p. 472, Lemma A.10.1) or Gantmacher (1986, pp. 411-412).

[2] For the notion of reducibility and irreducibility, see Emmenegger et al. (2020, p. 459) or, Gantmacher (1986, p. 417).

A Specific Sraffa Price Model With Subsistence Wages

Now, we formulate a specific Sraffa price model (Sraffa, 1960), see Emmenegger et al. (2020, pp. 110-118), where all the workers obtain exclusively subsistence wages, consisting of commodities like a bundle of agricultural products as wages. The conditions are adapted to the above discussed national IOTs. All the surplus goes as profits to the entrepreneurs. There is the semi-positive matrix $S \geq O$. There is a positive vector of surplus d $> o$, and therefore a positive vector of total output $q = Se + d > o$, as discussed for the Austrian IOT 2015. We set up the corresponding specific Sraffa price model, leading to an eigenvalue equation, using the coefficients matrix $C = S\hat{q}^{-1} \geq O$, which is reducible or irreducible as matrices Z,

$$\mathbf{S}'\mathbf{p}(1+R) = \hat{\mathbf{q}}\mathbf{p} = \mathbf{x} \quad \Leftrightarrow \quad \mathbf{C}'\mathbf{p}(1+R) = \mathbf{p} \quad \Leftrightarrow \quad \mathbf{C}'\mathbf{p} = \lambda_C \mathbf{p}. \tag{2}$$

Lemma A.10.3 (Emmenegger et al., 2020, p. 476) is applied. The Frobenius number of C' is positive, $\lambda_C = 1/(1+R) > 0$, as it has been stated for matrix Z. The special requests $x > o$ and $q > o$ for IOTs imply a positive price vector $p > o$. Equation $x = \hat{q}p > o$ holds. The value of the means of production is termed as *circulating capital* K $= e'(S'p)$, the value of the total output is X $= e'(\hat{q}p) = e'x$.

Every *CPA product group* is characterized by four specific *attributes*: quantity q_i, price p_i, value $x_i = p_i q_i$, and object $e_i, i = 1, \ldots, n$. These attributes can be combined to the following vectors: the vector of total outputs of quantities $q = [q_1, \ldots, q_n]' > o$, the price vector $p = [p_1, .., p_n]' > o$, the vector of values $x = \hat{q}p = [x_1, .., x_n]' > o$ and the vector of objects $e = [1, .., 1]'$, the objects giving the units of production for one period.

In the symmetric I-O-Tables (Table 1) the total outputs are equal to the total outlays, $x = y$. This is an accounting identity, an equilibrium between the input of the *means of production* of the product groups and the output of the *homogenous branches*, using the vectors of interindustrial products $x_I = Ze$, of *final use* $f > o$, of total *supply*, of intermediate inputs $y_I = Z'e$, together with the vector of value added v. This leads to the identity,

$$\mathbf{x} = \mathbf{Ze} + \mathbf{f} = \mathbf{x}_I + \mathbf{f} = \mathbf{y}_I + \mathbf{v} = \mathbf{Z}'\mathbf{e} + \mathbf{v} = \mathbf{y}. \tag{3}$$

The Leontief-type Input-Output Tables represent matrix components $z_{ij}, i, j = 1, \ldots, n$, periodic commodity flows in *monetary terms*. Piero Sraffa (1960) aimed commodity flows in physical terms. Sraffa proposed to represent by matrix components s_{ij}, quantities of commodities or bundles of commodities, analogue to CPA product groups[3] (Produktmix et al., 2011, p. 11), generating a commodity flow matrix S in *physical terms*, representing the means of production. With prices p_i the equations $z_{ij} = p_i s_{ij}$ hold[4]. An analogue relation exists between the *total final use* f_j and the *surplus* d_j, giving $f_j = p_j d_j$. The price vector $p > o$ is then diagonalized to the diagonal matrix $\hat{p}$. The equations are written in matrix form, giving $Z = \hat{p}S$ and $f = \hat{p}d > o$.

[3] in German: Produktmix et al. (2011, p. 11).

[4] If the bundles of commodities are subdivided down to the level of single products like wheat or coal, the physical terms become physical measure units like kilogram, liter, etc.

A Capital Productivity Measure: Productiveness

Using the above presented notions, we propose a measure of *capital productivity*, putting the value of the *annual surplus* d of an economy, as outputs, in relation to the value of the *means of production*, represented by matrix S, as inputs. We need an intermediate result: $d = q - Se = q - C\hat{q}e = Iq - Cq = (I - C)q$. Taking the price vector p, the value of the *volume of output* is obtained, $p'd = p'(I - C)q$. The *volume of total input* of production is the summed-up *commodity flow matrix in physical terms*, $Se = S(\hat{q}q) = (S\hat{q})q = Cq$, its value is the scalar $p'Cq$. We obtain as measure of *capital productivity*,

$$\rho = \frac{\mathbf{p}'\mathbf{d}}{\mathbf{p}'\mathbf{C}\mathbf{q}} = \frac{\mathbf{p}'(\mathbf{I} - \mathbf{C})\mathbf{q}}{\mathbf{p}'\mathbf{C}\mathbf{q}}. \tag{4}$$

Now we take the specific Sraffa price model (2), where all the surplus goes into the profits of the entrepreneurs, transposing, we get $p'C + Rp'C = p'I$, multiplying that equation from the right side with vector q, solving for R,

$$\mathbf{p}'\mathbf{C}\mathbf{q} + R\mathbf{p}'\mathbf{C}\mathbf{q} = \mathbf{p}'\mathbf{I}\mathbf{q} \Rightarrow R\mathbf{p}'\mathbf{C}\mathbf{q} = \mathbf{p}'\mathbf{I}\mathbf{q} - \mathbf{p}'\mathbf{C}\mathbf{q} \Rightarrow R = \frac{\mathbf{p}'(\mathbf{I} - \mathbf{C})\mathbf{q}}{\mathbf{p}'\mathbf{C}\mathbf{q}} = \rho. \tag{5}$$

The term, $R = \rho$, is the above defined *capital productivity* (4). We call R the *productiveness* of the economy[5]. Lemma 1 shows that the productiveness R of an economy (5) of a background Sraffa price model (2), is directly computed in terms of a boundary value problem defined on matrix Z, see also Emmenegger et al. (2020, pp. 399-408).

Lemma 1: Consider a reducible flow commodity matrix $Z = \hat{p}S$ with positive Frobenius number $\lambda_Z > 0$. Set up the vectors $y_I = Z'e,\ y = x = Ze + f$ and $v = y - y_I$. Compute a right eigenvector $s_1,\ Zs_1 = \lambda_Z s_1$. The productiveness R (5), associated to a background Sraffa price model (2), is the quotient of the scalar products $s_1'v$ and $s_1{}'y_I$. The Frobenius number $\lambda_C = \frac{1}{1+R}$ of C (2) is the quotient of the scalar products $s_1{}'y_I$ and $s_1{}'y$,

$$R = \frac{s_1{}'v}{s_1{}'y_I} \quad , \qquad \frac{s_1{}'y_I}{s_1{}'x} = \frac{s_1{}'y_I}{s_1{}'y} = \frac{1}{1+R} = \lambda_c \tag{6}$$

Proof: We start from the background Sraffa price model (2): $q = Se + d,\ S'p(1 + R) = \hat{q}p = y$, setting $C = S\hat{q}^{-1},\ \lambda_C := \frac{1}{1+R}$ => $S'p = \lambda_C\hat{q}p$. By definition: $Z' = S'\hat{p}$ and $y_I = Z'e = S'\hat{p}e = S'p$. Then, one gets with (2), (3) $y = \hat{q}p$: $(\lambda_F s_1')e = (s_1'Z')e = s_1'(Z'e) = s_1{}'y_I$. For this reason, one obtains: $\mathbf{s}_1'\mathbf{y}_I = \mathbf{s}_1'\mathbf{S}'\mathbf{p} = \mathbf{s}_1'\left(\lambda_C\hat{\mathbf{q}}\mathbf{p}\right) = \lambda_C\left(\mathbf{s}_1'\mathbf{y}\right) = \lambda_C\,\mathbf{s}_1'(\mathbf{y}_I + \mathbf{v})$.

$$\Rightarrow \lambda_C = \frac{\mathbf{s}_1'\mathbf{y}_I}{\mathbf{s}_1'\mathbf{y}} = \frac{\mathbf{s}_1'\mathbf{y}_I}{\mathbf{s}_1'\mathbf{y}_I + \mathbf{s}_1'\mathbf{v}} = \frac{1}{1 + (\frac{\mathbf{s}_1'\mathbf{v}}{\mathbf{s}_1'\mathbf{y}_I})} =: \frac{1}{1+R}, \text{ leading to } R = \frac{\mathbf{s}_1'\mathbf{v}}{\mathbf{s}_1'\mathbf{y}_I}. \ \blacktriangle$$

We illustrate Lemma 1 by an elementary computational example.

Example 1: Consider an economy consisting of $n = 2$ sectors, from which one knows an annual

[5] To our knowledge the term productivity of the economy (in German: Mass der Produktivität der Wirtschaft) for this number R has been proposed by Helmut Knolle in the year 2010 in a similar context of another specific Sraffa price model with irreducible matrix, a so-called standard system with standard commodity.

commodity flow matrix and the vector of supply in physical terms,

$$\mathbf{S} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}. \tag{7}$$

Compute the vector $q = Se + d > o$, the coefficients matrix $C = S\hat{q}^{-1} > O$, the Frobenius number $\lambda_C > 0$ of the matrix C, the productivity $R = (\frac{1}{\lambda_C}) - 1 > 0$ and the price eigenvector $p = [1, p_2]'$ of matrix C.

Then, compute the commodity flow matrix $Z = \hat{p}S$ in monetary terms and its Frobenius number λ_Z. Compute then the vectors x, y, x_I, y_I and v, as well as the normalized right eigenvector $s_1 = [s_{11}, 1]'$ of Z, $Zs_1 = \lambda_Z s_1$. Then, one verifies the equalities $\lambda_C = \frac{1}{1+R} = \frac{s_1' y_I}{s_1' y}$ and $R = \frac{s_1' v}{s_1' y_I}$.

Solution of Example 1: One gets,

$$\begin{aligned} \mathbf{q} = \mathbf{S}\mathbf{e} + \mathbf{d} &= \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix} \\ \mathbf{C} = \mathbf{S}\hat{\mathbf{q}}^{-1} &= \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{4}{5} \\ \frac{1}{9} & \frac{2}{5} \end{bmatrix}. \end{aligned} \tag{8}$$

Here, the Frobenius number of matrix C has been calculated, $\lambda_C = 2/3$, the normalized price eigenvector of C is $p = [1, 3]'$. We also compute the different requested vectors,

$$\begin{aligned} &\mathbf{Z} = \hat{\mathbf{p}}\mathbf{S} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 3 & 6 \end{bmatrix}, \quad \mathbf{f} = \hat{\mathbf{p}}\mathbf{d} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \\ &\mathbf{x}_I = \mathbf{Z}\mathbf{e} = \begin{bmatrix} 3 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}, \quad \mathbf{y}_I = \mathbf{Z}'\mathbf{e} = \begin{bmatrix} 3 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}, \\ &\mathbf{x} = (\mathbf{Z}\mathbf{e} + \mathbf{f}) = (\begin{bmatrix} 3 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix}) = \begin{bmatrix} 9 \\ 15 \end{bmatrix} = \mathbf{y}, \quad \mathbf{v} = \mathbf{y} - \mathbf{y}_I = \begin{bmatrix} 3 \\ 5 \end{bmatrix}. \end{aligned} \tag{9}$$

One computes the Frobenius number $\lambda_Z = 8.275$ of matrix Z and the associated normalized right eigenvector $s_1 = [0.758, 1]'$. Then,

$$\lambda_C = \frac{1}{1+R} = \frac{\mathbf{s}_1'\mathbf{y}_\mathbf{I}}{\mathbf{s}_1'\mathbf{y}} = \frac{\begin{bmatrix} 0.758 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 10 \end{bmatrix}}{\begin{bmatrix} 0.758 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 15 \end{bmatrix}} = \frac{2}{3}, \; R = \frac{\mathbf{s}_1'\mathbf{v}}{\mathbf{s}_1'\mathbf{y}_I} = \frac{\begin{bmatrix} 0.758 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix}}{\begin{bmatrix} 0.758 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 10 \end{bmatrix}} = \frac{1}{2}. \blacktriangle \tag{10}$$

Lemma 1 shows that the Frobenius number λ_C and the productiveness R are obtained *directly* as a solution of a *boundary value problem*, defined on matrix Z.

Applications to IOTs of Germany, Austria, and Switzerland

The National Statistical Offices publish commodity flow matrices Z in *monetary terms*. We have chosen three countries: Germany, Austria and Switzerland with national IOTs of different size and structure[6]. The three

[6] Statistisches Bundesamt, Deutschland, Input-Output Tabelle 2013 (Revision 2014, Stand: August 2017, Stand: 12.06.2018); Austrian IOT 2015 (with National Accounts, Version: May 2018), Swiss IOT 2014, Bundesamt für Statistik, Schweiz (Stand: 2018).

IOTs are structured schematically as the IOT of Table 1, a $n \times n$ *commodity flow matrix* Z, an $n \times m$ matrix of *total final use* and a $k \times n$ matrix of *value added* (in Table 1 *total final use* and *value added* are presented as vectors)[7], as presented in Table 2.

Table 2

Characteristics of National IOTs

Subject	Germany	Austria	Switzerland
Year	2013	2015	2014
Commodity flow matrix	72 × 72	65 × 65	49 × 49
No: (Laufende Nr.)			
Matrix of final use	72 × 12	65 × 15	49 × 24
Matrix of value added	10 × 72	16 × 65	4 × 49

One identifies the characteristic quantities of the elements of *national accounts*, obtainable form the IOTs. There is the *final consumption of households* C, the total *investments* I, the *government expenditures* G, the *exports* E and the *imports* M, Table 4. To these five characteristics correspond: the vector of final consumption of household $c = \{c_i\}$, the vector of total investments $i = \{i_i\}$, the vector of government expenditures $g = \{g_i\}$, the vector of exports $e_x = \{e_{xi}\}, i = 1, \ldots, n$, and the row vector of imports $m' = \{m_j\}, j = 1, \ldots, n$. We constitute the vector of total final use $f = c + i + g + (e_x - m')$, summing it up, one gets,

$$\mathrm{F} = e'f = e'(c + i + g + (e_x - m')) = \mathrm{C} + \mathrm{I} + \mathrm{G} + (\mathrm{E} - \mathrm{M}) \tag{11}$$

Table 4 contains the *inter-industrial circulating capital* K$= e'Ze$, the *total final use* of an economy F$= e'f$, the *total output* X = K + F and the GDP equal to Y = F - M.

Table 3

Official Aggregates of the National Accounts (Germany, A, CH)

Countries 1 EURO =1.15 CHF	Total Inter-ind. circulation capital K	Total final use F	Gross domestic product Y ~ GDP	Total output X
Germany (EURO)	2,831,297	3,875,317	2,826,240	6,706,614
Austria (EURO)	328,767	506,745	344,272	835,512
Switzerland (CHF)	662,275	932,706	649,718	1,594,980

The structure and the size of the three official national IOTs (Germany, A, CH) are different from each other (Table 2) as well as the content of their sectors. The main aggregates G, I, C, E, M, Y, computed form these national IOTs, are exactly the official values, published by the respective countries Germany, Austria, Switzerland for the indicated years. Then we compute the productiveness R of the economies (Table 4), using equation (6), Lemma 1, a capital productivity of a background Sraffa price model, which by definition measures the values in physical terms. This is a new measure of capital productivity and its deeper meaning and understanding will need more applications! It is known that measures of productiveness R are sensitive to the structure of the IOTs and that sector reduction influences the Frobenius number, see Emmenegger et al. (2020, pp. 425-427, Figure 10.12, Figure 10.3, Figure 10.14). The productiveness measure R changes whenever the

[7] If there are zero rows and columns belonging to the same sector, they are extracted from the IOTs. This is the case for the Austrian IOT 2015, but is not the case for the German IOT 2013, and the Swiss IOT 2014. Thus, there are positive output vectors in the three treated national IOTs as it is requested.

table of a specific country is aggregated. Also, we have not any information about the R-behaviour of Germany, Austria or Switzerland, when the tables themselves have different dimensions.

Table 4

Official Aggregates of National Accounts and the Productiveness

Countries	Final consumption (use) of commodities				Import M	Productiveness R
	Consump. of house-holds C	Total invest-ments I	Governm. expend. G	Export E		
Germany (EURO)	1,472,436	551,462	593,728	1,257,691	1,049,077	0.86281
Austria (EURO)	188,848	81,955	68,033	167,908	162,473	0.44157
Switzerland (CHF)	345,035	158,682	77,777	351,212	282,987	1.08840
1 EURO=1.15 CHF						

Conclusions

We consider a general Input-Output Table (IOT) in monetary terms, reflecting the production of that economy. We assume at the central part of that IOT a non-negative reducible commodity flow matrix Z, exhibiting a positive Frobenius number and positive Frobenius right eigenvectors s_1. A measure of capital productivity, called productiveness, is proposed. The productiveness R is defined as the value of the annually produced surplus of commodities and services, as outputs, divided by the value of the annually used means of production, as inputs, interpreted in physical terms, and results from the solution of a boundary value problem, defined on matrix Z. This productiveness R is a quotient, namely the scalar product of a right eigenvector s_1 of matrix Z and the vector of value added v, divided by the scalar product of s_1 and the vector of intermediate inputs y_I.

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