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Ostrowski's Method for Solving Nonlinear Equations and Systems

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Abstract: The dynamic characteristics and the efficiency of the Ostrowski's method allow it to be crowned as an excellent tool for solving nonlinear problems. This article shows different versions of the classic method that allow it to be applied to a wide range of engineering problems. Among them stands out the derivative-free definition applying divided differences, the introduction of memory and its extension to the resolution of nonlinear systems of equations. All of these versions are compared in a numerical simulations section where the results obtained are compared with other classic methods.

Key words: Iterative methods, nonlinear equations, convergence order, stability.

1. Introduction

Iterative methods arise to solve the problem of finding the zeros of functions whose nature is not linear. Numerical methods must find a balance between efficiency and accuracy. As far as possible, the method must commit to being computationally efficient and optimal, which gives it strength, robustness and a wide range of applications. The starting point of the problem of finding the zeros of a nonlinear function $f: I \subseteq R \to R$ that is, we set out to solve $f(x) = 0$.

Nowadays, there are a great variety of iterative methods that offer a range of exact and efficient solutions to this type of problem. Among all, the Newton's method stands out for its simplicity and efficiency.

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \tag{1}$$

A numerical method is said to be optimal if it keeps the balance between efficiency and the computational cost necessary to solve a given problem.

The order of convergence p of the method is defined in Refs. [1] and [2].

$$\lim_{k\to\infty} \frac{|x_{k+1} - \alpha|}{|x_k - \alpha|^p} = C \tag{2}$$

The efficiency index I of the method compares the order of convergence p with the number of functional evaluations per iteration d.

$$I = p^{1/d} \tag{3}$$

The error equation of a p-order method is defined by the following equation.

$$e_{k+1} = Ce_k^p + O\left(e_k^{p+1}\right) \tag{4}$$

2. Classic Ostrowski's Method

Multistep methods were developed to improve the local order of convergence of classic methods and thus to improve the efficiency index. Among these methods Ostrowski's method is found which is optimal in the sense of the Kung-Traub's conjecture.

Ostrowski's method belongs to King's family, which is defined in Ref. [3]:

$$y_k = z_k - \frac{f(z_k)}{f'(z_k)} \tag{5}$$

$$z_{k+1} = y_k - \frac{f(z_k) + (2+\beta)f(y_k)}{f(z_k) + \beta f(y_k)} \frac{f(y_k)}{f'(z_k)} \tag{6}$$

In order to obtain the iterative expression of the Ostrowski's method, the value of $\beta = -2$ must be replaced in the previous expression.

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)} \tag{7}$$

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$$x_{k+1} = y_k - \frac{f(x_k)}{f(x_k) - 2f(y_k)} \frac{f(y_k)}{f'(x_k)} \quad (8)$$

According to the following expression, Ostrowski's method has a convergence order four and therefore, according to the Kung-Traub conjecture, it is an optimal method since it uses three functional evaluations per iteration.

$$\text{Error} = (C_2^3 - C_2C_3)e^4 - 2(2C_2^4 - 4C_2^2C_3 + C_3^2 + C_2C_4)e^5 + O[e]^6 \quad (9)$$

This method carries out three functional evaluations per iteration therefore, it has an efficiency index $I \approx 1.5874$ higher than Newton's method.

In addition, according to the Kung-Traub's conjecture, it has the maximum order for its corresponding number of functional evaluations therefore, it is confirmed that Ostrowski's method is an optimal method.

3. Derivative-Free Ostrowski's Method

Ostrowski developed two methods of third and fourth order of convergence respectively and each of them requires the evaluation of two functions and one derivative per iteration.

Sometimes the functions are not differentiable at certain points that in most cases match with the solution points of the equation or simply its derivative requires a high computational cost, sacrificing the efficiency.

For solving this problem, the derivative-free iterative methods arise, these are based on a new definition of the derivative defined by the divided differences. Every iterative method can be transformed into a derivative-free method, but its order of convergence is not always preserved. The classic derivative-free method is Steffensen's method which is an optimal method too. This has the following iterative expression:

$$x_{k+1} = x_k - \frac{f(x_k)^{\,2}}{f\big(x_k + f(x_k)\big) - f(x_k)} \quad (10)$$

According to the following expression, the derivative-free Ostrowski's method defined by using divided differences of type $f[x_k, x_k + f(x_k)]$ has a convergence order three and therefore, according to the Kung-Traub's conjecture, it is no longer an optimal method.

Nevertheless, the derivative-free Ostrowski's method defined by using divided differences of type $f[x_k, x_k + f(x_k)^2]$ recovers fourth order of convergence and therefore the method is optimal again.

$$Error = C_2(-dFa^2C_2 + C_2^2 - C_3)e^4 + O[e]^5 \quad (11)$$

4. Ostrowski's Method with Memory

In order to increase the order of convergence, methods with memory arise which include in their iterative expression one or more iterations prior to the current one $x_k, x_{k-1}, \dots, x_{k-n}$.

The classic method with memory is Secant's method, which is defined by using divided differences of type $f[x_{k-1}, x_k]$ in Newton's method. According to the following expression, Secant's method performs a single functional evaluation per iteration, which increase the efficiency index of the method until 1.618.

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})} \quad (12)$$

Ostrowski's method defined by using an accelerant parameter γ has a convergence order three. Nevertheless, replacing the parameter $\gamma = -C_2$ it is possible to void the term e^3 and recover the fourth order of convergence.

$$Error = (-C_2^3 + 2C_2C_3)e^4 + (C_2^4 - 2C_2^2C_3 - 2C_3^2 + 3C_2C_4)e^5 + O[e]^6 \quad (13)$$

5. Dynamics of Ostrowski's Method

The stability of a method is a measure of the feasibility of initial estimates sets and it is considerate a discriminant factor in the selection process.

This section contains complex dynamics tools widely developed in Refs. [4] and [5].

In the study of King's family stability, the fixed points, the rational operator and all the analysis tools will depend on the parameter β, which allows studying the existence of a value of β that improves the stability of the rational operator, so it allows finding elements of the family that are more stable than others.

$$K_p(z,\beta) = z^4 \frac{5 + z^2 + 2\beta + z(4+\beta)}{1 + z(4+\beta) + z^2(5+2\beta)} \quad (14)$$

By looking at the exponent of this expression, it can be deduced that the order of convergence of this family is four and replacing $\beta = -2$ in the previous expression, the rational operator of Ostrowski's method is obtained.

$$K_p(z,-2) = z^4 \quad (15)$$

In order to obtain fixed points of King's family, the following equation must be solved.

$$K_p(z,\beta) = z \quad (16)$$

Once the previous equation is solved, the points $z = 0$ and $z = \infty$ which are conjugate fixed points are obtained.

Once the fixed points have been calculated, next step is to obtain the critical points by solving the equation $R'(z) = 0$. Thus, the critical points obtained are $z = 0$ and $z = \infty$. For $\beta = -2$, it is known that there are no free critical points, therefore, there are no critical points that are neither $z = 0$ or $z = \infty$. As in each basin there is a critical point and the only critical points are the roots, there are no basins different from the roots, this means that there is only one possible attractor behavior, the convergence of the roots. This feature provides the Ostrowski's method with excellent dynamic behavior.

The figure 1 shows the dynamic plane of Ostrowski's method in which the fixed points are represented with a white circle, the attractors with a star and the critical points with a square.

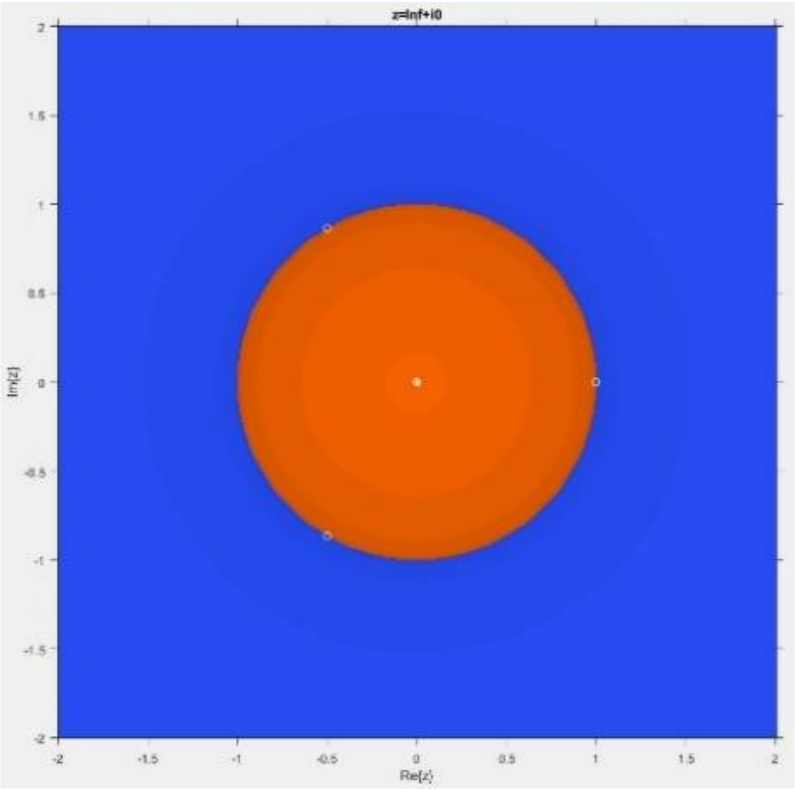

Fig. 1 Ostrowski's dynamic plane.

In this dynamic plane there are no black regions, which means that it is a stable method. This is because there are no critical points different from the two roots, then the only possible behavior is to converge to them. Only two basins of attraction (orange and blue) are observed, which correspond to these roots. Furthermore, there are two superattractors corresponding to $z = 0$ and $z = \infty$, which supports that the dynamic behavior of this method is stable. For instance, the orbit described by any point of the attraction basin is shown, where it can be seen that the orbit converges to the superattractive fixed point $z = 0$.

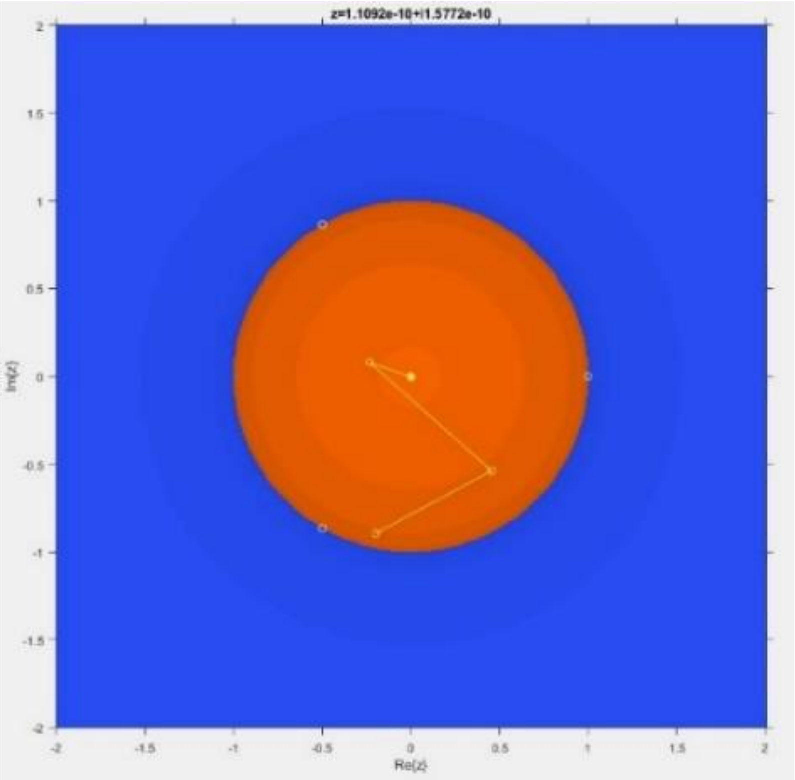

Fig. 2 Orbit.

6. Numerical Simulations

In this section, the Ostrowski method and its variants mentioned above are applied to the resolution of equations and nonlinear systems, comparing the obtained results with another classical methods. All results include their approximate computational order of convergence, usually called ACOC defined by Cordero and Torregrosa in Ref. [6].

$$p \approx ACOC = \frac{\ln\left(\frac{x_{k+1}-x_k}{x_k-x_{k-1}}\right)}{\ln\left(\frac{x_k-x_{k-1}}{x_{k-1}-x_{k-2}}\right)} \qquad (17)$$

The following cases will be analyzed:

- Find the zeros of two nonlinear functions applying classic Ostrowski's method and comparing the results obtained with Newton's method.
- Find the zeros of two nonlinear functions applying derivative-free Ostrowski's method and comparing the results obtained with Steffensen's method.
- Find the zeros of two nonlinear functions applying Ostrowski's method with memory and comparing the results obtained with Secant's method.
- Solve a nonlinear system of equations applying classic Ostrowski's method and comparing the results obtained with Newton's method.

The following two nonlinear equations will be solved to find its zeros on basis of begin point x_0 until solution $\tilde{x}$ is obtained with a tolerance of 10^{-8} in a i number of iterations.

$$f(x) = cos(x) - x$$
$$g(x) = x^3 - 10$$

In the first case, the results obtained by applying classic Ostrowski's method versus results obtained with Newton's method are shown in Table 1.

Table 1 Classic Ostrowski versus Newton.

Classic Ostrowski				
Function	x_0	$\tilde{x}$	i	ACOC
$f(x)$	1	0,7391	3	3,4628
$g(x)$	2	2,1544	4	4,0445
Newton				
Function	x_0	$\tilde{x}$	i	ACOC
$f(x)$	1	0,7391	4	1,9988
$g(x)$	2	2,1544	7	2,0007

In the second case, the results obtained by applying derivative-free Ostrowski's method defined by two types of divided differences versus results obtained with Steffensen's method are shown in Table 2.

Table 2 Derivative-free Ostrowski versus Steffensen.

Ostrowski $f[x_k, x_k + f(x_k)]$				
Function	x_0	$\tilde{x}$	i	ACOC
$f(x)$	1	0,7391	3	2,9729
$g(x)$	2,1	2,1544	4	3,0248
Ostrowski $f[x_k, x_k + f(x\_k)^2]$				
Function	x_0	$\tilde{x}$	i	ACOC
$f(x)$	1	0,7391	3	4,0049
$g(x)$	2,1	2,1544	3	5,1469
Steffensen				
Function	x_0	$\tilde{x}$	i	ACOC
$f(x)$	1	0,7391	4	1,9999
$g(x)$	2,1	2,1544	6	2,0002

In the third case, the results obtained by applying Ostrowski's method with memory versus results obtained with Secant's method are shown in the following table.

Table 3 Ostrowski with memory versus Secant.

Ostrowski with memory					
Function	x_{-1}	x_0	$\tilde{x}$	i	ACOC
$f(x)$	-1	0	0,7391	4	3,0072
$g(x)$	1	2	2,1544	3	2,9826
Secant					
Function	x_{-1}	x_0	$\tilde{x}$	i	ACOC
$f(x)$	-1	0	0,7391	7	1,6205
$g(x)$	1	2	2,1544	6	1,5221

Last case the following $2x2$ nonlinear system is solved and the results obtained by applying Classic Ostrowski's method versus results obtained with Newton's method are shown in the following table.

$$\begin{cases} y = x^2 - 2x + 1 \\ y = -2x^2 - 3x + 1 \end{cases} \tag{18}$$

Table 4 Classic Ostrowski versus Newton.

System of equations				
Method	x	y	i	ACOC
Newton	1	0	8	1,9835
Ostrowski	1	0	7	1,9957

7. Conclusions

Ostrowski's method is optimal in the sense of the Kung-Traub's conjecture.

Ostrowski's method has an excellent dynamic behavior because it does not have critical points different from the two roots, then the only possible behavior is to converge to them.

In this dynamic plane of Ostrowski's method there are no black regions, which means that it is a stable method. It only has two basins of attraction (orange and blue) which correspond to the roots. Furthermore, there are two superattractors corresponding to $z = 0$ and $z = \infty$, which supports that the dynamic behavior of this method is stable.

About the results obtained in above cases where Ostrowski's method and its variants have been compared with another classical methods, from the same starting point all methods converge to the same solution. However, Ostrowski's method is more efficient since it uses fewer iterations to obtain the same result, so its computational cost is less than Newton's method.

Derivative-free methods need greater precision with starting point estimation to avoid convergence problems. For instance, in function $g(x)$ the initial point has been adjusted to $x_0 = 2.1$ whereas in classic Ostrowski's method or Newton's method it is enough to estimate the initial point at $x_0 = 2$.

It can be seen that the ACOC coefficient is close to the order of convergence of each method, taking into account that the order of convergence is theoretical and the ACOC is nothing more than a numerical approximation. Nevertheless, in the case of Ostrowski's method with memory it is observed that the ACOC obtained is one order lower than should theoretically be obtained using the accelerator parameter γ. This is a clear example that introducing accelerators does not always preserve the order of convergence of the classic method, less if the derivatives are replaced by divided differences. Therefore, to preserve the order of convergence the key is a great quality with derivatives approximation.

In the case of systems of equations, ACOC either preserves the order of convergence of the classic Ostrowski's method, since the order of convergence of this method is 4 and in the analyzed system it hardly reaches the order 3.

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