Development and Background of Shells Theories—A Review Study

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Abstract: Thin shells are one of the structural elements that have versatile contributions in different engineering sectors, specifically in architectural, civil, mechanical, aeronautical, and marine engineering industries. Liquid-retaining structures, wide-span roofs, water tanks, arch dooms, and shells used in building nuclear power plants are recognized application examples of shell structures in architectural and civil engineering. This variety in using shells in different engineering sectors is due to the productivity of load-carrying behavior, excellent reservation in strength and structural integrity, shell structures are preferable in comparison to structural systems having the same span and dimensions; high stiffness, and covering a large areas. Besides the above distinguishing mechanical pros, it is widely accepted that structures and building containing shells are usually preferred by architectures and designers for aesthetic purposes. The analysis of shells has gone through many stages until the arrival of modern theories. In this study the different theories of shells were discussed, the background and development of shell theories were illustrated in this investigation.

Key words: Shell, analysis, stress, Hooke's law, elasticity.

1. Introduction

Thin shells are one of the structural elements that have versatile contributions in different engineering sectors, specifically in architectural, civil, mechanical, aeronautical, and marine engineering industries. Liquid-retaining structures, wide-span roofs, water tanks, arch dooms, and shells used in building nuclear power plants are recognized application examples of shell structures in architectural and civil engineering. This variety in using shells in different engineering sectors is due to the following merits:

(1) Productivity of load-carrying behavior.

(2) Excellent reservation in strength and structural integrity.

(3) High strength relative to self weight, this standard is usually adopted to predict structural component effectiveness: the higher this ratio, the more optimal structure.

(4) Relatively high stiffness.

(5) Cover large spaces.

Besides the above distinguishing mechanical properties, it is widely accepted that structures and building containing shells are usually preferred by architectures and designers for aesthetic purposes [1].

2. Types of State of Stress for Thin Shells

It was already supposed, as the general theory of linear shell was being stated, that the values of flexural stresses have similar order as those stresses resulted from stresses that are in-plane (membrane). Below are typical expected cases thin shells stress states.

(1) If the flexural stresses are too small relative to the membrane ones then such a stress state type is called a membrane or moment-less state of stress. It is reported that membrane theory sometimes has the capability of representing the stresses and strains developed in shells with rational precision. One reason for that is the negligible value of twisting and bending moments. For instance, if a shell of hollow spherical geometry is applied to uniform outside and

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inside pressure, it is considered under pure membrane state of stress.

(2) The membrane stresses are insignificant with respect to the flexural stresses, and then such a type of state of stress is denoted as a pure flexural or moment state of stress. Consider that the definitions of the "membrane" and "pure flexural" states of stress are not completely precise as the membrane state of stress admits an existence of small flexural stresses and, in turn, small membrane stresses might happen in a pure flexural state of stress. As illustrated earlier, the relatively small thickness of shells makes their flexural stiffness limited. Hence, their resistance capacity to flexural bending is quite low. As a result, shells could endure large displacement and flexural stresses just because of comparatively small applied bending moments. That is, subjecting shells to pure flexural stresses is quite risky, actually this is one of the main drawbacks of the shells when used as structural members. Several treatments have been proposed to deal with this drawback: (i) utilizing adequate intermediate reinforcements, (ii) choosing adequate support type and (iii) selecting corresponding shell shape. The membrane state of stress, in contrast, is the most preferred stress state. As at this state, the shell experiences uniformly stresses along its thickness. Engineers and designers, as a consequence, should do their best and come up with structural designs that as much as possible ensure membrane state of shell stress.

(3) If the membrane and flexural stresses are the same, a state of stress is called a mixed state, or edge effect. The term "edge effect" is linked with the fact that the above-mentioned mixed state of stress usually happens close to the shell edges. It is worthwhile mentioning that only few attempts to end up with membrane stress state over the surface of a shell were successful, the majority were failed trials. A state of stress that is nearly membrane has been achieved at the main portion of a shell. In contrast, the mixed state of stress (edge effect) was the governing state near shell edges or around the reinforcing stiffeners. It was found that this state of stress usually happens in relatively small area. This stress analysis and description is beneficial in identifying between the two main shell stress states, membrane and mixed [1].

3. Historical Background of the Shell Theory

The linear elasticity approach was the key underlying basis for almost all the frequently used theories of shells. It is agreed that shell theories based on linearity have the ability of computing deformations and stresses values with acceptable precision. That is specifically true when the value of elastic deformation generated into the shell is relatively small. That is the deformations are small enough to apply the conditions of equilibrium equation as the shell surface is not deformed. In other words, the deformation obeys Hooke's law.

For the ease of analytic computations, and in order to utilize the principles of linear elasticity, a shell should be considered as 3-dimensional body. Nevertheless, computations relied on linear elasticity concept are generally not easy and time consuming. As a consequence, an alternative approach that is much easier and straightforward was developed in the theory of shells. The new simplified method involved presuming certain assumptions that convert a 3D analysis of shell into merely computing stresses and deformations of a two-dimensional body (thin shell). That is analyzing only the shell's middle surface as illustrated previously. In other words, the simplification adopted in the thin shell theory is primarily based on analyzing the deformations occurring in the shell's middle surface only.

A spectrum of theories for analyzing shells have been developed and presented in the literature; the amount of simplifying hypothesis determines the amount of difficulty in computations. Several related studies have discussed the required equilibrium between the approximations needed for easing the analysis on one side and the degree of calculations

accuracy on the other side. The following paragraphs attempt to highlight the main elastic shell theories that have been developed and published with a chronological order.

Love [2] conducted the pioneering work that utilized the traditional linear elasticity concept to propose an effective simplification for shell theory. Love succeeded in simplifying the mathematical models that link strain with-is placement and thus the essential relations to the theory of shells. Another approximation was suggested by Kirchhoff which was particularly customized to handle the plate bending theory. It involves assumptions regarding thin shell concept and small deflection. The Kirchhoff-Love assumptions are the term that involves all the above mentioned assumptions. The theory of thin shells based on linear elasticity principles proposed by Love has been known as First-order approximation. Although Love's theory has gained positive feedback and wide popularity, it has received some critics and few drawbacks were highlighted. One of the important drawbacks is its unclear approach in treating with small terms; even when they have the same order the theory procedure would reject some of them while accepting the rest for no obvious criterion. Technically, this implies that when setting up the equations of equilibrium for particular shells, the Love's differential operator matrix regarding displacements would be unsymmetrical. This was appraised as a clear contravention to the theorem of reciprocity proposed by Betti. Other deficiencies in the Love's theory have been also reported. A key one is that using Kirchhoff-Love approximation procedure to yield robust 2D set of shell equation has ended up with first order simplification theories that are dissimilar.

E. Reissner [3] also proposed thin shell theory based on linearity on the purpose of modifying some Love's theory drawbacks (also the first-order approximation theory). Based on the first-order approximation theory and the Love-Kirchhoff hypotheses, he succeeded in utilizing the three-dimensional theory of elasticity to build a derivation for strain-displacement relations, equations of equilibrium, and also for expressions regarding stress resultants. It is useful to underline that the proposed procedure involved discarding small terms of order $z = Ri$ (for $i = 1, 2$) where the term Ri represents the curvature radius of the middle shell surface with respect to unity in the equivalent terms.

Another work that was based on the Kirchhoff-Love initial assumptions was conducted by Sanders [4]; the work involved building new simplified shell theory that is first-order and based on the concepts of virtual work. Sanders' thin-shell theory can be considered as a step forward as it presented an effective mechanism to override the mismatching issue in the original shell theory presented by Love. A version that keeps terms of magnitude in consistence with those kept by E. Reissner and effectively handles the inconsistency challenge in the twist expression was suggested by Koiter [5].

Following, Timoshenko [6] has proposed a shell theory that is quite similar to the Love's one. In that, he developed a theory to derive the required relationships and equations based on the same simplifications proposed by Kirchhoff-Love thin elastic shell approach. In his procedure the term $z = Ri$ was also discarded when being small with respect to unity.

Naghdi [7, 8] appraised to what extent the Love-Kirchhoff's thin shell hypotheses are robust and with realistic results. Naghdi took the analysis into higher step by developing thin-elastic shells where higher-order approximation was derived and presented. In his works, there is a possibility to allocate all shell theories that are based on linearity assumption in which one or more assumptions of the Kirchhoff-Love theory are suspended. At the beginning, specific typical theories are adopted in which the thinness simplification is delayed in computations while the

other assumptions are reserved. This analytic scenario implies that the order of a certain approximate approach should be set up based on the sort of the terms in the thickness coordinate that is kept in the strain relations and constitutive-equations.

A new line of theory of shells with 2nd-order approximation was independently investigated by Lur'ye [9], Fliigge [10], and Byrne [11]. With respect to the relations and stress-related equations derived in this theory, they were totally based on the assumptions previously presented by Kirchhoff in addition to the hypothesis of small-deflection regarding the equivalent equations of the 3D theory of elasticity.

Further sophisticated approach is followed to ignore the term $z = Ri$ in the second-order approximation approach. That is, the approach attempts to keep these terms in relative with unity in the stress resultant equations and strain-displacement general relationships. The second-order simplification adopted in this shell theory has been successfully applied to analyze circular cylindrical shells. Moreover, the applicability of the developed strain-displacement relations and stress equations based on this second-order approximation theory was appraised as complicated and not straightforward.

A similar work was presented by Novozhilov [12] where second order approximation for the shell theory was presented. The analytic procedure was quite similar to that adopted in Lur'ye-Flugge-Byrnework. Novozhilov utilizing Kirchhoff's approach to work the 3D-theory of elasticity to get the general relationships related to the strains and displacements. Next, the procedure continues by taking advantage of Kirchhoff's simplification and strain-energy expression to develop both stress resultants and couples formulas and to identify the ignorable terms. In general, Novozhilov made use of Kirchhoff's assumptions to present analytic procedure to simplify the primary relations and equations of the general thin shell theory. The main finding of his work is that the deficiency in applying Kirchhoff's hypothesis in the

small-deflection shell theory is in the order $h = Ri$ with regard to unity.

The work conducted by Gol'denveizer [13] regarding the modification of thin shells' theory is highly recognized and appreciated. The work was the first in the line of theory of shells whereby specific compatibility conditions for the strain components were proposed. Another proposal for working out the original 3D elasticity equations for analyzing thick shells was made by Vlasov [14]. The proposal involved converting these three-dimensional equations into two dimensional ones to simplify the computations. For thin shell analysis, the simplifying hypothesis involves ignoring the transverse shear and normal components of the strain. The rest of strains that will be retained are only mathematically expressed by the first three terms of their corresponding series. The simplification approach adopted by Valsov by taking the values of transverse strain as zero has paved the way for smoothness moving from three- to two-dimensional thin plate equations.

Alternative approaches of second-order approximation mechanisms, in which both normal stress and transverse shear were taken into account, were examined by several researchers including E. Reissner [15, 16], and Naghdi [17]. Their hypothesis involved considering zero value of the transverse normal strain while keeping the normal. Other groups of researchers such as Kraus [18], Leissa [19], and others have carried out a thorough analysis of the first- and higher-order approximation regarding the thin shell elastic theories. They took into account the relevant general equations.

All their developed procedures regarding the small-defection shell theories have been originated based on the traditional theory of linear elasticity. The theories developed follow the law of Hooke and neglect the nonlinearity parts in strain and equilibrium equations. Such equations based on such postulates are mathematically having a unique solution for every

case. That is, the shell theories based on linear elasticity identify unique position of equilibrium for specific shell constraints and loads. Nevertheless, real-life problems of physical shell do not have unique solution on every occasion. That is, there is more than one expected position for equilibrium for a certain thin shell with certain cases of constraints and loadings. Thin shell theories are considered as nonlinear if they are considered finite or large deformations in their analytic derivations. The status of stress-strain relationship may make a thin shell as physically nonlinear.

A shell theory that involves nonlinearity was presented by E. Reissner [20] with symmetrically loaded shells of revolution. The developed theory does not consider the simplification regarding small-deflections; however, other simplifying postulates relevant to the universal higher-order approximation theories were taken into consideration. Researchers such as Naghdi and Nordgren [21], Sanders [22], and Koiter [23] have taken further step by introducing derivations of thin shell theories that consider further geometrically nonlinearity. A similar work that developed a shell theory with taking nonlinearity into account was carried out by Vlasov [14]. He specifically presented a group of equations for shallow shells Von Karman equations will be described. These equations can be considered a special case for the equations developed by Vlasov. Von Karman equations state finite deformations for plates. The complementary works of Mushtari and Galimov [24] and Simmonds and Danielson [25] can be considered as a continuation in the field of non-linear thin shell theories.

It is worthwhile mentioning that besides the works that proposed general shell theories, several studies have tackled the issue of developing thin shell theories for special cases where only certain kinds of shells have been adopted. For these specialized theories, the variety of simplifying hypotheses used and the scope of applicability expected are shaped by several factors.

Examples include ranges of deformation, the geometry of thin shell, the favorite behavior of shell required, stress condition and loading status. The first-order approximation proposed by Love shell theory was the underlying derivative basis for all these theories.

One pioneering and recognized example of these specialized thin shell theories is the membrane theory. During the previous century, Beltrami [26] and Lecornu [27] have derived and presented the overall form of the primary thin shell equations of membrane theory. Ever since, there have been rapid advancements in this theory. H. Reissner [28] derived the equations for thin shell membrane theory considering the case of unsymmetrical loading. Next, Sokolovskii [29] worked successfully of minimizing the membrane theory equations to canonical form and his work exposed many of their distinguishing properties. The merit of employing Airy's function of stress to enhance the results of membrane theory was investigated by Pucher [30]. His work was able to analyze shells of arbitrary form. Shallow shells are another case of the specialized shell theory.

Several works were separately carried out on the purpose of ending up with universal formula for thin shell theory that is less complicated. Examples of these works are Donnel [31], Vlasov [14], and Mushtari [32]. The resulting equations were quite simplified and appropriate enough to be adopted to tackle a wide spectrum of thin shell cases that were previously quite complex to be analyzed. Besides Kirchhoff-Love hypotheses, several other simplifying assumptions have been adopted in developing and deriving the equations and relations regarding stress, strain and compatibility. The findings and applications have obviously indicated the suitability of the Donnel-Vlasov-Mushtari theory in the field of what are so-called shallow shells. This sort of shells is geometrically quite similar to thin plates. That is why they are sometimes referred to as curved plates. They have been widely used in building and construction

industry; specifically, in constructing roofs whose rise is relatively small with respect to their lengths and widths. It is reported that the theory of Donnel-Vlasov-Mushtari can be utilized for analyzing shells that are not shallow as long as the components of stress along the middle surface coordinates change quickly.

Marguerre in his study has analytically developed the general equations and relations for plates with initial curvature [33]. Another case of specialized thin shell theories is the theory developed for shells of revolution. This group of shells is vital as it is widely used in a variety of engineering-related applications.

A traditional mathematical representation for bending problems for the shell of revolution theory was developed by H. Reissner [28] where he investigated a spherical shell subjected to axisymmetric bending. In his work, the asymptotic method was employed for integrating spherical shell's differential equations after simplifying them to a more appropriate form. Following, the findings of Reissner were developed to more general formula by Meissner [34]. That is by including shells of revolution with even deformation and of random thickness and shape.

A step forward in deriving shell theories was made by Hoff [35] when he considered the analysis of shells subjected to non-uniform applied loads and of circular conical geometry. A more general formula for such circular conical shaped shells was presented by Flugge where even nonsymmetrical loading can be involved [36]. The classical displacement approach was the analytic kernel of his proposed method.

Shells of toroidal type were considered by Wissler [37]. It is common that cylindrical shells and shells of revolution are usually analyzed separately despite the fact that they are technically classified under the same class of shells.

There has been a growing body of studies that dealt with the analysis of cylindrical shells subjected to wide range of loading cases. That is most probably due to their wide commercial use in addition to comparative straightforwardness of their underlying analytic theory. It is important to recall that the fundamental differential equations presented by Flugge for shells of circular cylindrical shapes were all related to displacements [36].

Parkus presented the general equations for a cylinder with nonuniform cross section [38]. Major simplifying assumptions were proposed and adopted by Donnel [31], Dishinger [39], Hoff [40], and Vlasov [14] in order to derive the main equations and relations related to the general cylindrical shell theory. Novozhilov [16] in his study derived the main governing formulas for cylindrical shells of non-uniform shape. Complex variables were involved and he highlighted a number of adequate simplifications for the developed equations.

As mentioned earlier, Love [2] was the first who presented the edge effect concept which was employed in the shell of revolution analyses. This concept has gained wide interest and popularity due to its power in easing the thin shell's engineering analysis. Love stated that a preliminary approximation to the general governing equation can be accomplished by considering jointly the sum of the solutions of both edge effect and membrane equations. Integrals of the edge effect equations can be written in the form of rapidly varying functions. Gol'denveizer [13] in his work stated the conditions required for the rapidly decaying solutions of the general shell equations to be attained, in addition, he examined several probable special cases. Taking this concept into consideration, Geckeler [41] studied the case of spherical shells subjected to symmetrical loading. The developed simplified technique depended on the decrease of the two-coupled differential equations system into only two independent differential equations with the assumption that for shells that are significantly thin the derivatives of the original functions are of higher order than the functions themselves.

A considerable amount of research has been recognized in the literature regarding the buckling

issue in shells. For instance, Lorentz [42] and Timoshenko [43] have derived the primary equations and presented the subsequent solutions for the case of circular cylindrical shell with uniform compressive axial loading. Following, the linear buckling issue generated within shells of cylindrical shapes was tackled by Mises [44] and Mushtari and Sachenkov [45]. The shells were loaded with axially compressive forces together with external pressure. A related work on cylindrical shell was done by Flugge where the combined effect of axial and internal pressure was examined [10].

It is quite central to underline that in contrast to the plate bending problems, buckling load, computed theoretically by the small-deflection concept, is very difficult and hardly to be achieved in real models. There is evidence that early deficiencies are the key principal reasons to the incongruity between theoretical and experimental findings in computing the buckling loads.

Taking this challenge into account, Donnel and Wan [46] proposed the cylindrical shell general equations and evaluated the influence of these early imperfections on the buckling behavior. The considered applied loads were axial compressive forces of uniform distribution. Similar works that considered the buckling issue for a spherical shell subjected to an external pressure were carried out by Zoelly [47] in linear and by von Karman and Tsien [48] in nonlinear analytic approach, correspondingly.

The post-buckling effects developed in cylindrical shell loaded by axial compressive force were investigated by Koiter [49]. In the same line of research, the buckling behavior with normal surface loading shallow shells was studied by Budansky [50] with linear formulation and by Kaplan and Fung [51] with nonlinear formulation.

On the other hand, the motion formulas needed for the vibration analysis of shells can be directly obtained as a straight forward generalization of the static equation by incorporating the inertia forces to the terms of body forces and body moment that yield from the accelerations of the shell mass as stated by the corresponding concept of D'Alambert.

4. Conclusions

The results of this paper contributing to the evaluation of development of shell theory. At least four conclusions can be achieved. Firstly, the study shows that the numerical approximation of a shell element can be adopted using the Donnel and Wan, but differences in the numerical results to closed-form Zoelly shell theory must be expected. Analytical solutions to the mathematical model given in the paper of Geckeler should ideally be used in the convergence studies. Secondly, the numerical analysis of shell elements based on degenerating three dimensional continuums to shell behavior should be performed using the mathematical model established and analyzed in the paper of Vlasov. Third, the higher order equation theory of Naghdi is more accurate than 2nd order analysis proposed by Lur'ye, Fliigge, and Byrne. Fourth, all discussed theories depend mainly on Hooke's law, Timoshenko proposed more reliable equations for shells analysis theory depending on Hooke's law.

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