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Arbitrage and Pricing in Financial Markets with Interval Data*

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Financial data are often affected by uncertainty: imprecision, incompleteness etc.. Uncertain data may be represented by intervals. Intervals may be useful for representing uncertainty in financial data or, by converse it may be useful to construct intervals from scalar financial data, for analyzing the uncertainty in the solution of real financial problems. Considering this different form of input data, a review of some financial models and definitions has been necessary. The notion of arbitrage is crucial in the modern theory of finance. It is the cornerstone of the option pricing theory. Roughly speaking a market is arbitrage-free if there is no way of making risk less profits. How to extend this definition when the returns are intervals? In the present work the definition of a system of returns which does not allow arbitrage opportunities is given for the case of interval returns. It is proved that, given a two-period economy $T = (t_0, t_1)$ and n securities, the system of returns at time t_0 which does not allow interval arbitrage opportunities, is an interval vector. Furthermore using the IntervalCAPM (Interval Capital Asset Pricing Model) methodology, in the present work the region of the plane, risk vs. expected return, where surely there are arbitrage opportunities is described. Some numerical results are presented: the interval beta and the interval alpha of the asset ABBOT (Abbot Laboratories), which belongs to the SP500 (Standard and Poor's 500 Composite) index, is estimated using the IntervalCAPM approach. The used algorithm has been implemented in MATLAB. The solutions obtained are always well interpretable.

Keywords: interval algebra, interval-valued variables, interval financial returns, interval arbitrage

Introduction

Financial data are often affected by uncertainty: imprecision, incompleteness etc.. Therefore, in a decision-making problem, we should be able to process uncertain information. The uncertainty in the data may be treated by considering, rather than a single value, the interval of values in which the data may fall. For example, many times we do not know the exact value of the return of an asset in the ith state of the world but we bet, at best, on the interval of its possible values. Intervals may be useful for representing uncertainty in financial data or, by converse it may be useful to construct intervals from scalar financial data, for analyzing the uncertainty in the solution of real financial problems. In section 3, as an advancement with

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respect to (Gioia, 2009): (1) The definition of a system of returns which does not allow arbitrage opportunities is given for the case of interval returns; and (2) It is proved that, given a two-period economy $T = (t_0, t_1)$ and n securities, the system of returns at time t_0 which does not allow interval arbitrage opportunities, is an interval vector. The CAPM is a pricing methodology, and as such it is supposed to provide a pricing functional for several assets that at least in the limit satisfy the no-arbitrage condition. In section 4, as an improvement of the IntervalCAPM methodology already known in the literature (Gioia, 2009) the region of the plane, risk vs. expected return, where surely there are arbitrage opportunities is described. In section 5 a real case is presented: the interval beta and the interval alpha of the asset ABBOT (Abbot Laboratories), which belongs to the SP500 (Standard and Poor's 500 Composite) index, is estimated using the IntervalCAPM approach. The downloaded data refer to single-valued variables; to manage uncertainty we have transformed these variables into interval-valued ones by applying a perturbation using a uniform distribution $U(0, 0.01)$.

Definitions Notations and Basic Facts

Let I be the set of closed and bounded intervals. If ● is one of the symbols +, -, / , · ,we define an arithmetic on I by:

$$[a,b]\bullet[c,d]=\{x\bullet y\mid \quad a\le x\le b, c\le y\le d\} \tag{1}$$

Except that we do not define $[a, b]/[c, d]$ if 0 is in $[c, d]$.

Definition 1. An interval-valued variable $\boldsymbol{X}_j^I$ is a variable which assumes an interval of values on each of k considered individuals: $\boldsymbol{X}_j^I = (X_{ij} = [\underline{x}_{ij}, \overline{x}_{ij}]), \quad i = 1,\cdots,k$.

Proposition 1. If $f(x_1, \ldots, x_n)$ is a real rational function in which each variable x_i occurs only once and only at the first power, then the corresponding interval expression $f(X_1, X_2,\ldots,X_n)$ will compute the actual range of values of f:

$$f(X_1,X_2,\ldots,X_n)=\{\, y \mid y=f(x_1,x_2,\ldots,x_n),\quad x_i\in X_i=[\underline{x}_i, \overline{x}_i],\ i=1,\ldots,n\}$$

The proof of this proposition may be found in Moore (1966).

Definition 2. A $k \times n$ interval matrix $\boldsymbol{A}^I$ is the following set of matrices:

$$\boldsymbol{A}^I = \{\, \boldsymbol{A} \mid \underline{\boldsymbol{A}} \le \boldsymbol{A} \le \overline{\boldsymbol{A}} \,\}$$

where $\underline{\boldsymbol{A}}$ and $\overline{\boldsymbol{A}}$ are $k \times n$ scalar matrices which verify $\underline{\boldsymbol{A}} \le \overline{\boldsymbol{A}}$. The inequalities are understood to be componentwise.

These results and other properties of the interval algebra may be found in Moore (1966), Alefeld and Herzberger (1983), and Alefeld and Mayer (2000).

Interval Arbitrage

Suppose we have a two period economy $T = (t_0, t_1)$ and n securities denoted as S_j ($j = 1, \ldots, n$). Let us indicate with $\boldsymbol{R^0} = (R_1^0, R_2^0, \cdots, R_n^0)$ the vector of the returns of the n securities at time t_0 and with x_j the portion of total investment funds devoted to this security. Thus $\sum_{j=1}^n x_j = 1$, where $x = (x_1, x_2, \ldots, x_n)^t$ is the portfolio of the considered consumer and X is the set of possible portfolios. At time t_1 the returns of the n securities are assumed to be $k \times 1$ random variables denoted by: $\boldsymbol{R}_j$ ($j = 1, \ldots, n$) (k different states of the world are contemplated), represented as columns of the following $k \times n$ matrix:

$$\boldsymbol{R} = \begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ R_{21} & R_{22} & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{k1} & R_{k2} & \cdots & R_{kn} \end{pmatrix}$$

The vector $p = (p_1, p_2, \ldots, p_k)$ is a discrete probability distribution on the outcomes of the random variable $\boldsymbol{R}_j$ ($j = 1, \ldots, k$). A market is *arbitrage-free* if there is no way of making riskless profits (Schachermayer, 2008), more formally:

Definition 3. A system of returns $\boldsymbol{R}^0$ at time t_0 does not allow arbitrage opportunities if and only if for all portfolios x:

$$\boldsymbol{R^0 x} \leq 0, \quad \boldsymbol{Rx} \leq 0 \tag{2}$$

For example an arbitrage opportunity would be a trading strategy which started at time t_0 with zero value, and terminated at time t_1 with a positive value. Financial data are often affected by uncertainty: imprecision, incompleteness etc. Therefore in setting the arbitrage-free returns of a set of securities, we should be able to process uncertainty. For example, let us suppose now that the returns of the security S_j ($j = 1, \ldots, n$) at time t_1 are represented by intervals of values when each state occurs. Thus $\boldsymbol{R}_j$ ($j = 1, \ldots, n$) are assumed to be interval-valued variables denoted by: $\boldsymbol{R}_j^{\boldsymbol{I}}$ ($j = 1, \ldots, n$) and represented as columns of the following interval matrix:

$$\boldsymbol{R}^I = \begin{bmatrix} [\underline{R}_{11}, \overline{R}_{11}] & \ldots & [\underline{R}_{1j}, \overline{R}_{1j}] & \ldots & [\underline{R}_{1n}, \overline{R}_{1n}] \\ \vdots & & \vdots & & \vdots \\ [\underline{R}_{i1}, \overline{R}_{i1}] & \ldots & [\underline{R}_{ij}, \overline{R}_{ij}] & \ldots & [\underline{R}_{in}, \overline{R}_{in}] \\ \vdots & & \vdots & & \vdots \\ [\underline{R}_{k1}, \overline{R}_{k1}] & \ldots & [\underline{R}_{kj}, \overline{R}_{kj}] & \ldots & [\underline{R}_{kp}, \overline{R}_{kp}] \end{bmatrix}$$

where $[\underline{R}_{ij}, \overline{R}_{ij}]$ is the interval in which the return rate of security S_j "falls" when the i^{th} state occurs.

Definition 4. A system of returns $\boldsymbol{R}^0$ at time t_0 does not allow interval arbitrage opportunities if and only if for all portfolios $\boldsymbol{x}$ in X:

$$\boldsymbol{R^0 x} \leq 0, \quad \boldsymbol{R}^I \boldsymbol{x} \leq 0 \tag{3}$$

where the operations are interval algebra operations, "≤" is an order relationship (Gioia, 2011) on the set of closed and bounded intervals $\boldsymbol{I}$ and the inequalities are understood to be component wise.

Proposition 2. The system of returns $\boldsymbol{R}^0$, which does not allow interval arbitrage opportunities, is an interval vector.

Proof. Let us suppose that the system of returns $\boldsymbol{R}^0$ does not allow arbitrage opportunities, thus equation (3) holds true; in it, *the interval system of inequalities* $\boldsymbol{R^I x \leq 0}$ (explicitly):

$$\begin{array}{c} [\underline{R}_{11}, \overline{R}_{11}]\, x_1 + [\underline{R}_{12}, \overline{R}_{12}]\, x_2 + \cdots + [\underline{R}_{1n}, \overline{R}_{1n}]\, x_n \leq 0 \\ [\underline{R}_{21}, \overline{R}_{21}]\, x_1 + [\underline{R}_{22}, \overline{R}_{22}]\, x_2 + \cdots + [\underline{R}_{2n}, \overline{R}_{2n}]\, x_n \leq 0 \\ \vdots \qquad \vdots \qquad \ddots \qquad \vdots \\ [\underline{R}_{k1}, \overline{R}_{k1}]\, x_1 + [\underline{R}_{k2}, \overline{R}_{k2}]\, x_2 + \cdots + [\underline{R}_{kn}, \overline{R}_{kn}]\, x_n \leq 0 \end{array} \tag{4}$$

is the set of systems of linear inequalities $\boldsymbol{Rx} \leq \boldsymbol{0}$ for all $\boldsymbol{R} \in \boldsymbol{R}^{\boldsymbol{I}}$. Let $\boldsymbol{x} \in X$ be a solution (Rohn, 1989) of equation (4), then:

$$\boldsymbol{Rx} \leq \boldsymbol{0}, \quad \forall \boldsymbol{R} \in \boldsymbol{R}^{I}$$

For hypothesis the system of returns $\boldsymbol{R}^0$ satisfies the inequality $\boldsymbol{R^0x} \leq 0$ thus for the Minkowsky-Farkas Lemma, for all $\boldsymbol{R}$ in $\boldsymbol{R}^I$ it exists $y(\boldsymbol{R}) = (y_1, y_2, \ldots, y_k) \in R_+^k$ so that:

$$\begin{array}{l} y_1 R_{11} + y_2 R_{21} + \cdots + y_k R_{k1} = R_1^0 \\ y_1 R_{12} + y_2 R_{22} + \cdots + y_k R_{k2} = R_2^0 \\ \vdots \qquad \vdots \quad \ddots \quad \vdots \\ y_1 R_{1n} + y_2 R_{2n} + \cdots + y_k R_{kn} = R_n^0 \end{array} \tag{5}$$

The expressions on the left side of equation (5) are real rational functions indicated as:

$$f_j(R_{1j}, \cdots, R_{kj}),\ j = 1, \cdots, n \tag{6}$$

in which R_{ij} ($i = 1, \ldots, k$) occur only once and at the first power, then for Proposition 1 if we substitute to R_{ij} the interval $[\underline{R}_{ij}, \overline{R}_{ij}]$, the corresponding interval expression: $f_j^I = f_j([\underline{R}_{1j}, \overline{R}_{1j}], \ldots, [\underline{R}_{kj}, \overline{R}_{kj}]), \quad j = 1, \cdots, n$ computes the actual range of each f_j, that is the set of values of function (6) varying each variable R_{ij} in its own interval of variation, explicitly:

$$\begin{array}{l} f_1^I = y_1[\underline{R}_{11}, \overline{R}_{11}] + y_2[\underline{R}_{21}, \overline{R}_{21}] + \cdots + y_k[\underline{R}_{k1}, \overline{R}_{k1}] := [\underline{R}_1^0, \overline{R}_1^0] \\ f_2^I = y_1[\underline{R}_{12}, \overline{R}_{12}] + y_2[\underline{R}_{22}, \overline{R}_{22}] + \cdots + y_k[\underline{R}_{k2}, \overline{R}_{k2}] := [\underline{R}_2^0, \overline{R}_2^0] \\ \vdots \qquad \vdots \quad \ddots \quad \vdots \\ f_n^I = y_1[\underline{R}_{1n}, \overline{R}_{1n}] + y_2[\underline{R}_{2n}, \overline{R}_{2n}] + \cdots + y_k[\underline{R}_{kn}, \overline{R}_{kn}] := [\underline{R}_n^0, \overline{R}_n^0] \end{array} \tag{7}$$

thus at time t_0 the system of returns which does not allow arbitrage opportunities is the following interval vector:

$$\boldsymbol{R^0} = \left([\underline{R}_1^0, \overline{R}_1^0], [\underline{R}_2^0, \overline{R}_2^0], \cdots, [\underline{R}_n^0, \overline{R}_n^0]\right)$$

When the returns of some risky assets at time t_1 are not known precisely in each state of the world (are intervals), also the non-arbitrage system of returns at time t_0 is subject to uncertainty and, for Proposition 2, it comes out that it is itself an interval vector. In fact, the interval system of returns which does not allow arbitrage opportunities may be regarded as the set of systems of scalar returns which do not give rise to any possibility of arbitrage. On the contrary, a system of returns which would give rise to arbitrage opportunities is any vector of scalar returns not belonging to $\boldsymbol{R^0}$.

Interval CAPM and Arbitrage Opportunities

The Capital Asset Pricing Model (CAPM) (Sharpe, 1964; Eichberger & Harper, 1997) is a pricing methodology, and as such it is supposed to provide a pricing functional for several assets that at least in the limit satisfy the no-arbitrage condition. In the case of *interval returns*, the interval CAPM has been treated in (Gioia, 2009) in the present section, we aim at describing, in this special case, the region of the plain, risk versus expected return, where surely there are arbitrage opportunities. It is well known the *Security Market Line* (SML):

$$E(\boldsymbol{R}_j) = r_f + (E(\boldsymbol{A}) - r_f) \cdot \beta_j \tag{8}$$

where $E(\boldsymbol{R}_j)=\sum_{i=1}^{k} p_i R_{ij}$, $E(\boldsymbol{A})=\sum_{i=1}^{k} E(\boldsymbol{R}_j)A_j$, are the expected return of the j^{th} security and of the market portfolio respectively; r_f is the risk-free rate and the factor of proportionality β_j has the following expression: $\beta_j=\sigma(\boldsymbol{A},j)/\sigma^2(\boldsymbol{A})$ having indicated with $\sigma(\boldsymbol{A}, j)$ the covariance between the return of the market portfolio and the return of the j^{th} asset, and with $\sigma^2(\boldsymbol{A})$ the variance of the market portfolio. Let $\boldsymbol{Z}_{jt}$ and $\boldsymbol{Z}_{mt}$ (Campbell, Lo, & MacKinlay, 1997) be the vectors of the excess returns of the asset S_j and the market portfolio respectively in different periods of time, those excess returns can be described using the excess-return single-index marked model:

$$\boldsymbol{Z}_{jt}=\alpha+\beta\,\boldsymbol{Z}_{mt}+\boldsymbol{e}_{jt} \tag{9}$$

where $\boldsymbol{e}_{jt}$ is the vector of disturbances and the hypotheses concerning time-independence are supposed to hold true. It is known from classical theory that estimators for β and α are the OLS (Ordinary Least Square) estimators. The IntervalCAPM introduced in (Gioia, 2009) applies when the returns are described by interval-valued variables and all the necessary assumptions for equations (8) and (9) are required to hold true. It uses the interval algebra operations directly in equation (8) to compute: (1) the interval expected *return* of asset S_j, when the interval expected return of the market portfolio and the interval beta are known; (2) the *interval beta*, when the interval expected return of asset S_j and of the market portfolio are known. Furthermore, when interval time-series of the returns in equation (9) are available, the IntervalCAPM algorithm computes the interval slope of the Security Market Line by means of an interval regression technique (Gioia, 2005) based on an optimization method; in particular IntervalCAPM computes: the interval $\hat{\beta}_j^I$ and the interval $\hat{\alpha}_j^I$ which are the sets of all β and α of relation (9) respectively, when each return R_{ij}, $i = 1, \dots, k$ ranges in its own interval of values; the interval expected return of asset S_j is then computed using equation (8) and combining the obtained intervals with the interval algebra operations. It is important to notice that IntervalCAPM makes extensively use of the interval algebra tools combined with some optimization techniques to consider the interval as a whole structure and to compute the interval of solutions, which is the interval containing all possible values assumed by a considered function when the observed values vary in their own interval of values. The CAPM states that the equilibrium price of a risky asset given its beta, is the point on the Security Market Line corresponding to that beta; each point not belonging to the Security Market Line creates arbitrage opportunities. In the case of interval returns, the beta is an interval thus we may consider, rather than the equilibrium price of the risky asset, its interval equilibrium price. Let us give a graphical interpretation: when the returns vary in their intervals of variation, the value of the market portfolio, depending on that returns, also varies and in particular its expected excess return ranges in the interval $E^I(\boldsymbol{A})=\left[\underline{E}(\boldsymbol{A})-r_f,\overline{E}(\boldsymbol{A})-r_f\right]$; the equation (8) becomes:

$$E(\boldsymbol{R}_j)=\left[\underline{E}(\boldsymbol{A})-r_f,\overline{E}(\boldsymbol{A})-r_f\right]\beta_j+r_f \tag{10}$$

By means of the interval algebra operations, equation (10) describes in the plane (β, $E(\boldsymbol{R})$) a sheaf of straight lines of intercept r_f and slope in $E^I(\boldsymbol{A})$-r_f as described in Figure 1. Let us consider the risky asset S_j, its interval beta $\hat{\beta}_j^I=[\underline{\beta}_j,\overline{\beta}_j]$ is represented on the horizontal axis; a β_j in $\hat{\beta}_j^I$ identifies a segment which contains the set of equilibrium returns of S_j, with respect to β_j, for each value of the slope $E(\boldsymbol{A})$ in its interval of values $E^I(\boldsymbol{A})$. When β describes the whole interval $\hat{\beta}_j^I$, region $\boldsymbol{B}$ in Figure 1 is represented:

$$\boldsymbol{B} = \left\{(\beta, E(\boldsymbol{R}_j)) : \beta_j \in \hat{\beta}_j^I,\ E(\boldsymbol{A}) \in E^I(\boldsymbol{A}),\ E(\boldsymbol{R}_j) = (E(\boldsymbol{A}) - r_f)\beta_j + r_f\right\}$$

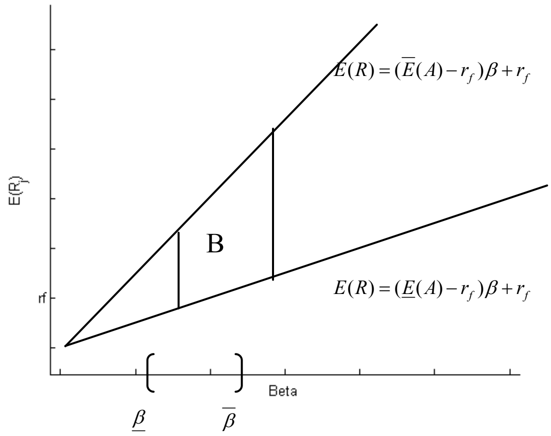

Figure 1. Region **B**.

Furthermore, when the returns of asset S_j are not known precisely, even the corresponding β_j and $E(\boldsymbol{R}_j)$ are unknown punctually, $\boldsymbol{A}$ describes the set of possible combination of β_j and $E(\boldsymbol{R}_j)$ for each value of the returns each of which in its range of variations. The interval expected return of the asset S_j at time t_0 is computed as:

$$\left[\underline{E}(\boldsymbol{R}_j), \overline{E}(\boldsymbol{R}_j)\right] = \left[\underline{E}(\boldsymbol{A})\underline{\beta}_j + r_f\ ,\ \overline{E}(\boldsymbol{A})\overline{\beta}_j + r_f\right]. \tag{11}$$

which contains the set of all equilibrium returns of S_j as each R_{ij} at time t_1 varies in its interval of variation. It is known that the CAPM is a pricing methodology, and as such it is supposed to provide a pricing functional for S_j, which is in practice never exactly correct because for example, as in this special case, the returns of the asset at time t_1 are known approximately, i.e., are intervals; a point not on the Security Market Line is considered an "arbitrage point" but maybe it is not. In the interval case, the uncertainty in the returns is taken into account considering the interval returns, and a set containing all possible equilibrium returns (region **B**) is computed: each point not region **B** surely allows arbitrage opportunity. The IntervalCAPM should be regarded as a method which may give some additional information to that provided by the classical method, in fact the classical CAPM computes the equilibrium return of S_j, the IntervalCAPM computes the "uncertainty" around this value. Let us point out that one of the advantages in using the described methodology for interval data is to estimate how a perturbation on the input data (returns) reflects on the final solution. In fact each interval solution must be regarded as a set of solutions of infinitely different problems generated by the values of the quantity involved (returns) each of which in its own interval of variation. For example the interval expectations involved in equations (10) and (11) are the sets containing the set of all expectations of the excess return of the market portfolio and of the returns of security S_j respectively, for each value of the return R_{ij} in $[\underline{R}_{ij}, \overline{R}_{ij}]$. In conclusion, interval methods aim at enforcing the power of decision of classical methods analyzing the interval of solutions when each quantity varies in its own interval of values.

Real Case

In the following example the interval risk and the interval expected return of the asset ABBOT, which belongs to the SP500 (Standard and Poor's 500 Composite) index, are estimated using the IntervalCAPM approach. The forecast of some interval returns of asset ABBOT, referring to different periods of time, is also computed (see Table 1). The downloaded data refer to single-valued variables; to manage uncertainty we have transformed these variables into interval-valued ones by applying a perturbation using a uniform distribution $U(0, 0.01)$. The interval β^I (4th column in Table 1) is well interpretable considering that it does not contain the zero; an investor knows that, even if the returns fluctuate around their fixed values, the β is always positive and it ranges from 0.060 to 1.007. Furthermore, remarking that for a single-valued security the CAPM states that the intercept α in equation (9) is zero, the interval α^I is interpretable as the set of all "errors" that we may do using the CAPM for predicting the expected return of the considered risky security. It is remarkable that α^I = [-0,044, 0,0005] is an interval around the zero which does not contain elements with absolute value significantly different from zero, thus IntervalCAPM may be a good way to make forecast. Finally constructing on a Cartesian plane the region **B** described in Section 4, with respect to the computed β^I and $E(\boldsymbol{R})^I$ (5th column in Table 1), an investor knows which are the points in the plain that surely allow arbitrage opportunities.

Table 1
Interval CAPM on Asset ABBOTT

Asset: ABBOTT	Forecast	α^I	β^I	$E(\boldsymbol{R})^I$
[0.005, 0.014]	[-0.043, 0.021]	[-0.044, 0.005]	[0.060, 1.007]	[-0.023, 0.027]
[-0.023, -0.017]	[-0.081, 0.003]			
[0.000, 0.009]	[-0.106, 0.002]			
[-0.043, -0.039]	[-0.068, 0.004]			
[-0.053, -0.041]	[-0.157, 0.000]			
[-0.058, -0.046]	[-0.064, 0.004]			

Conclusions and Future Perspectives

In this paper a special kind of data is considered: the interval data. Intervals may be useful for representing the uncertainty in the data or by converse, it may be useful to construct intervals, from scalar financial data, for aggregating huge number of data; furthermore intervals may be used to analyze how a perturbation on the input data reflect on the final solution. Considering this different form of input data a review of some financial models and definitions has been necessary. In this paper: (1) the definition of a system of returns which does not allow arbitrage opportunities is given for the case of interval returns; (2) it is proved that, given a two-period economy $T = (t_0, t_1)$ and n securities, the system of returns at time t_0 which does not allow interval arbitrage opportunities, is an interval vector. The CAPM is a pricing methodology, and as such it is supposed to provide a pricing functional for several assets that at least in the limit satisfy the no-arbitrage condition. One of the drawbacks of the CAPM is that it provides a pricing functional for an asset S_j which is in practice never exactly correct because for example, the returns of S_j at time t_1 are known approximately; a point not on the Security Market Line may be considered an "arbitrage point" but maybe it is not. In the case of uncertain returns, the interval CAPM (IntervalCAPM) has been treated in (Gioia, 2009). In this paper as an improvement with respect to (Gioia, 2009), the region of the plane, risk vs. expected return, where surely there are arbitrage opportunities is described; a region containing all possible equilibrium returns is computed: each point not in that region "surely" allows arbitrage opportunity. Some numerical results are presented: the interval beta and the interval alpha of the asset ABBOT (Abbot Laboratories), which belongs to the SP500 (Standard and Poor's 500 Composite) index, is estimated using the IntervalCAPM approach. The forecast of some interval returns of the considered assets, referring to different periods of time, is also computed. The used algorithm has been implemented in MATLAB. The solutions obtained are always well interpretable. As a future prospective of research it could be interesting to give a demonstration of the Fundamental Theorem of Asset Pricing in the special case of interval data.

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