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Approximation Properties For Modified Kantorovich-Type Operators

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Abstract: We introduce a modification of Kantorovich-type operators in polynomial weighted spaces of functions. Then we study some approximation properties of these operators. We give some inequalities for these operators by means of the weighted modulus continuity and also obtain a Voronovskaya-type theorem. Furthermore, in our paper show that the operators give better degree of approximation of functions belonging to weighted spaces than classical Szász- Kantorovich operators.

Keywords: Kantorovich-type operators, modules of continuity, weighted spaces, Voronovskaya theorem.

1. Introduction

In 1930 Kantorovich operators $K_n := L_1[0,1] \to C[0,1]$ were introduced by the following operators:

$$K_n(f)(x) = (n+1)\sum_{k=0}^{\infty}\binom{n}{k}x^k(1-x)^{n-k}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(s)\,ds,\ n \in N \tag{1}$$

(see [4]) where $f \in L_1[0,1]$ and $x \in [0,1]$. Clearly, Kantorovich operators are linear and positive. Note that Kantorovich operators are extension of classical Bernstein operators in order to study the approximation in the integrable function space $L_1[0,1]$. Inspried by these operators many authors studied Kantorovich extensions of some linear positive operators, some are in [1,6,9,12] and references therein. In the last decade, these kinds of researches have been continuied.

In 1978 Becker [2] studied some approximation problems of Szász-Mirakyan operators.

$$S_n(f)(x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right),\ x \in R_0 = [0,\infty) \tag{2}$$

for $f \in C_p$, where C_p with fixed $p \in N_0 := \{0,1,2,...\}$ denotes the polynomial weighted space generated by the weight function

$$w_0(x) := 1,\ w_p(x) := \left(1+x^p\right)^{-1},\ p \ge 1, \tag{3}$$

i.e. C_p is the set of all real-valued functions f continuous on R_0 and such that $w_p f$ is uniformly continuos and bounded on R_0. The norm in C_p is defined by the formula

$$\|f\|_p := \|f(.)\|_p := \sup_{x \in R_0} w_p(x)\left|f(x)\right|. \tag{4}$$

In [2,10], the degree of approximation of $f \in C_p$ by the operators (2) were proved. It was proved that

$$\lim_{n\to\infty} S_n(f)(x) = f(x) \tag{5}$$

for every $f \in C_p, p \in N_0$ and $x \in R_0$. Moreover, the convergence in 5 is uniform on every interval $[x_1, x_2], x_2 > x_1 \ge 0$. Then, Z.Walczak made some works. In [11] Walczak considered the space

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$C_p^1(x) := \{f \in C_p : f' \in C_p\}$ and defined the following modulus of continuity $w_1(f; C_p; t)$ for $f \in C_p$

$$w_1(f; C_p; t) := \sup_{0 \le h \le t} \left\| \Delta_h f(.) \right\|_p, \forall t \in R_0 \quad (6)$$

where $\Delta_h f(x) := f(x+h) - f(x)$ for $h, x \in R_0$. Therefore

$$\lim_{t \to 0^+} w_1(f; C_p; t) = 0, f \in C_p.$$

Morever, if $f \in C_p^1$ then there exists a positive constant M such that $w_1(f; C_p; t) \le M_1 t$ for $t \in R_0$. He introduced the following operators: Let $p \in N_0, r \in N$ be fixed numbers. For $f \in C_p$,

$$A_n^*(f; r; x) := \frac{1}{g((nx+1)^2; r)} \sum_{k=0}^{\infty} \frac{(nx+1)^{2k}}{(k+r)!} f\left(\frac{k+r}{n(nx+1)}\right) \quad (7)$$

where

$$g(t; r) = \sum_{k=0}^{\infty} \frac{t^k}{(k+r)!} \quad (8)$$

and

$$g(0; r) = \frac{1}{r!}, \; g(t; r) = \frac{1}{t^r}\left(e^t - \sum_{j=0}^{r-1} \frac{t^j}{j!}\right), t \in R. \quad (9)$$

Szász-Mirakyan Kantorovich operators is defined as follow;

$$T_n(f)(x) = ne^{-nx} \sum_{k=0}^{\infty} \binom{n}{k} x^k \frac{(nx)^k}{k!} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\, dt \quad (10)$$

for $x \in R_0, p \in N_0 := \{1, 2, 3, ...\}, f \in L_1[0, \infty)$. Some modification of the operators may be viewed in [3,4,5,8]. In this work, we consider a Kantorovich-type modification of the operators (7) and obtain the results of Walzack [11] for these operators. Also we study convergence properties of these operators for functions in C_p and C_p^1.

2. Construction of the Operators

Definition 1. Let $p \in N_0$ and $r \in N$ be fixed numbers and (a_n) be a positive sequence such that $\lim_{n \to \infty} a_n = \infty$. For functions $f \in C_p$, we introduce the operators

$$A_n(f; r; x) := \frac{a_n}{g((a_n x+1)^2; r)} \sum_{k=0}^{\infty} \frac{(a_n x+1)^{2k}}{(k+r)!} \int_{\frac{k+r}{a_n}}^{\frac{k+r+1}{a_n}} f\left(\frac{t}{a_n x+1}\right) dt, \quad (11)$$

where (8)-(9) hold.

Linearity and positivity of the operator A_n are clear. Also, we see easily that the following holds;

$$\frac{1}{g(t; r)} \le r!. \quad (12)$$

We shall prove that A_n is an operator from C_p into C_p for every fixed $p \in R_0$. In this paper, we use notation $g_{n,r}(x)$ instead of $g((a_n x+1)^2; r)$. The moments are obtained as follow:

$$A_n(1; r; x) = 1, \quad (13)$$

$$A_n(t; r; x) = \left(x + \frac{1}{a_n}\right)\left(1 + \frac{1}{2(a_n x+1)^2} + \frac{1}{(a_n x+1)^2 (r-1)! g_{n,r}(x)}\right), \quad (14)$$

$$A_n(t^2; r; x) = \left(x + \frac{1}{a_n}\right)^2 \left[1 + \frac{2}{(a_n x+1)^2} + \frac{1}{3(a_n x+1)^4} + \frac{1}{(a_n x+1)^2 (r-1)! g_{n,r}(x)}\left(1 + \frac{r+1}{(a_n x+1)^2}\right)\right], \quad (15)$$

$$A_n\left(t^3;r;x\right)=\left(x+\frac{1}{a_n}\right)^3\left[1+\frac{3}{(a_nx+1)^2}+\frac{5}{3(a_nx+1)^4}+\frac{1}{4(a_nx+1)^6}\right.$$
$$\left.+\frac{1}{(a_nx+1)^2(r-1)!g_{n,r}(x)}\left(1+\frac{2r+7}{2(a_nx+1)^2}+\frac{2r^2+3r+2}{2(a_nx+1)^4}\right)\right] \quad (16)$$

for every fixed numbers $r,n\in N$ and $x\in R_0$.

3. Main Results

We can prove following the Lemmas by using (11)-(16).

Lemma 1. Let $n\in N$ be fixed. Then for all $x\in R_0$, we have

$$A_n\left(\left(t-x\right);r;x\right)=\frac{1}{a_n}+\frac{1}{2a_n(a_nx+1)}+\frac{1}{a_n(a_nx+1)(r-1)!g_{n,r}(x)}, \quad (17)$$

$$A_n\left(\left(t-x\right)^2;r;x\right)=\frac{2}{a_n^2}+\frac{(r-1)!g_{n,r}(x)+3(r+1)+3(a_nx+1)^2}{6a_n^2(a_nx+1)^2(r-1)!g_{n,r}(x)}$$
$$-\frac{x}{2a_n^2(a_nx+1)}+\frac{-x(a_nx+1)^3(r-1)!+a_n^2x^2(a_nx+1)^2}{a_n^2(a_nx+1)^2(r-1)!}+\frac{(a_nx+1)^4(r-1)!}{a_n^2(a_nx+1)^2}, \quad (18)$$

$$A_n\left(\left(t-x\right)^3;r;x\right)=\frac{2}{a_n^3}+\frac{2\left(a_n^2+a_n+1\right)(a_nx+1)^3\left(r-1\right)!g_{n,r}\left(x\right)+2r^2+3r+2}{2a_n^3(a_nx+1)^3\left(r-1\right)!g_{n,r}\left(x\right)}$$
$$+\frac{4(a_nx+1)+(2r+7)(a_nx+1)^2}{2a_n^3(a_nx+1)^3}+\frac{6(a_nx+1)^3+a_n^3(a_nx+1)^2}{2a_n^3(a_nx+1)^3}-\frac{6x(a_nx+1)}{2a_n^3(a_nx+1)^3}$$
$$-\frac{x}{a_n^3(a_nx+1)^2}-\frac{3x(r+1)}{a_n^3(a_nx+1)^2\left(r-1\right)!g_{n,r}\left(x\right)}+\frac{-12a_n^3\left(r-1\right)!g_{n,r}\left(x\right)-3xa_n}{2n^3\left(r-1\right)!g_{n,r}\left(x\right)} \quad (19)$$
$$+\frac{-3x(a_nx+1)+3x^2a_n^2}{a_n^3(a_nx+1)^2\left(r-1\right)!}+\frac{3x(a_nx+1)^2}{(a_nx+1)\left(r-1\right)!g_{n,r}\left(x\right)}-\frac{x}{a_n^3(a_nx+1)^2}+\frac{3x^2}{a_n(a_nx+1)}.$$

Lemma 2. Let $r,s\in N$ be fixed Then there exist positive numbers $\alpha_{s,j}$ depending only j,s,γ_j and $\beta_{s,j}\left(r\right)=r^{j-1}$, depending only on j and s, $1\le j\le s$ such that

$$A_n\left(t^{s+1};r;x\right)=\left(x+\frac{1}{a_n}\right)^{s+1}\left\{\sum_{j=1}^{s+1}\frac{1}{(a_nx+1)^{2(j-1)}}\left(\alpha_{s,j}+\frac{\gamma_j}{(a_nx+1)^2}+\frac{\beta_{s,j}\left(r\right)}{(a_nx+1)^2\left(r-1\right)!g_{n,r}\left(x\right)}\right)\right\} \quad (20)$$

for all $n\in N$ and $x\in R_0$ where $\alpha_{s,s}=\alpha_{s,1}=\gamma_1=1,\beta_{s,1}\left(r\right)$ and $\alpha_{s,j},\gamma_j$ are constants.

Proof. We prove this lemma by using the methods and results of Lemma 2 in [11]. From (13) and (16) we see that (20) is obtained for $s=0,1,2$. Let (20) holds for $f_j\left(x\right):=x^j,1\le j\le s$ with fixed $s\in N$. We shall prove (20) for $f_j\left(x\right)=x^{s+1}$. From (7), (11) and (12) it follows that

$$A_n\left(t^{s+1};r;x\right)=\frac{1}{(s+2)a_n^{s+1}(a_nx+1)^{s+1}}+\frac{\sum_{i=1}^{s+1}\binom{s+2}{i}r^{s+1-i}}{(s+2)a_n^{s+1}(a_nx+1)^{s+1}(r-1)!g_{n,r}(x)}$$

$$+\frac{1}{a_n^{s+1}(a_nx+1)}+\frac{\sum_{i=1}^{s}\binom{s+2}{i}}{(s+2)a_n^{s+1}(a_nx+1)^{s+1}}\times\sum_{l=1}^{(s+1)-i}\binom{(s+1)-i}{l}\left(a_nx+1\right)^l a_n{}^l A_n^*\left(t^l;r;x\right).$$

Using results of Lemma 2 in [11] and taking the assumption into account, we get that

$$A_n\left(t^{s+1};r;x\right)=\frac{1}{(s+2)a_n^{s+1}(a_nx+1)^{s+1}}+\frac{\sum_{i=1}^{s+1}\binom{s+2}{i}r^{s+1-i}}{(s+2)a_n^{s+1}(a_nx+1)^{s+1}(r-1)!g_{n,r}(x)}$$

$$+\frac{1}{a_n^{s+1}(a_nx+1)}+\frac{\sum_{i=1}^{s}\binom{s+2}{i}}{(s+2)a_n^{s+1}(a_nx+1)^{s+1}}\sum_{l=1}^{(s+1)-i}\binom{(s+1)-i}{l}$$

$$\times(a_nx+1)^{2l}\sum_{j=1}^{l}\frac{1}{(a_nx+1)^{2(j-1)}}\left(\alpha_{l,j}+\frac{\beta_{l,j}\left(r\right)}{(a_nx+1)^2\left(r-1\right)!g_{n,r}\left(x\right)}\right)$$

$$=\left(x+\frac{1}{a_n}\right)^{s+1}\left\{\frac{1}{(a_nx+1)^{2s}}\left(1+\frac{\sum_{i=1}^{s+1}\binom{s+2}{i}r^{(s+1)-i}}{(s+2)(a_nx+1)^2\left(r-1\right)!g_{n,r}\left(x\right)}\right)\right.$$

$$+\frac{\sum_{i=1}^{s}\binom{s+2}{i}}{(s+2)(a_nx+1)^{2s}}+\frac{1}{(s+2)(a_nx+1)^{2(s+1)}}+\sum_{j=1}^{s}\sum_{l=2-j+1}^{s}\frac{1}{(a_nx+1)^{2(j-1)}}$$

$$\left.\times\left(\alpha_{l,j}+\frac{\beta_{l,j}\left(r\right)}{(a_nx+1)^2\left(r-1\right)!g_{n,r}\left(x\right)}\right)\right\}.$$

Hence we have the desired result of (20).

Lemma 3. Let $p\in N_0$ and $r\in N$ be fixed. Then there exists positive constants $M_2=M_2\left(p,r\right)$ and $M_3=M_3\left(p,r\right)$ depending only on the parameters p and r such that

$$\left\| A_n\left(\frac{1}{w_p(t)};r;.\right)\right\|_p \le M_2, n\in N \tag{21}$$

and for all $f\in C_p$, we have

$$\left\| A_n(f;r;.)\right\|_p \le M_3\|f\|_p, n\in N. \tag{22}$$

Proof. For $p=0$, we get $A_n(f_0;x,y)=1$. Let $p\in N$ be fixed. From (11)-(16) we have

$$w_p(x)A_n\left(\frac{1}{w_p(t)};f;r;x\right)=w_p(x)\left\{1+A_n\left(t^p;f;r;x\right)\right\}$$

$$=\frac{1}{1+x^p}+\frac{\left(x+\frac{1}{a_n}\right)^p}{1+x^p}\sum_{j=1}^{s}\frac{1}{(a_nx+1)^{2(j-1)}}\left(\alpha_{s,j}+\frac{\gamma_j}{(a_nx+1)^2}+\frac{\beta_{s,j}(r)}{(a_nx+1)^2(r-1)!g_{n,r}(x)}\right)$$

$$\le 1+p\left(\alpha_{s,j}+\frac{\gamma_j}{(a_nx+1)^2}+\frac{\beta_{s,j}(r)}{(a_nx+1)^2(r-1)!g_{n,r}(x)}\right)\le M_2(p;r).$$

Thefore, we obtain

$$\left|w_p(x)A_n\left(\frac{1}{w_p(t)};f;r;x\right)\right|\le\frac{a_n}{w_p(x)g_{n,r}(x)}\sum_{k=0}^{m}\frac{(a_nx+1)^{2k}}{(k+r)!}\int_{\frac{k+r}{n}}^{\frac{k+r+1}{n}}w_p(x)\left|f\left(\frac{t}{(a_nx+1)}\right)\right|dt,$$

which gives the desired result. For (22) we have the following inequalities;

$$A_n\left(\frac{w_p(t)}{w_p(t)}f;r;x\right)=A_n\left(\sup\left(fw_p(t)\right)\frac{1}{w_p(t)};r;x\right)\le\|f\|_pA_n\left(\frac{1}{w_p(t)};r;x\right).$$

Therefore, using (21), we get (22).

Lemma 4. Let $p\in N_0$ and $r\in N$ be fixed. Then there exists a positive constant $M_4=M_4(p,r)$ depending only on the parameters p and r such that

$$\left\| A_n\left(\frac{(t-.)^2}{w_p(t)};r;.\right)\right\|_p \le\frac{M_4}{a_n^2}, n\in N. \tag{23}$$

Proof. For $p=0$ the formulas given in Lemmas 1-3 and (11) imply (23). By (3) and (13)-(16) we have

$A_n\left((t-x)^2/w_p(t);r;x\right)=$

$A_n\left((t-x)^2;r;x\right)+A_n\left(t(t-x)^2;r;x\right)=$

$A_n\left((t-x)^3;r;x\right)+(1+x)A_n\left((t-x)^2;r;x\right)p,$

$n\in N.$

If $p=1$ then

$A_n\left((t-x)^2/w_1(t);r;x\right)=$

$A_n\left((t-x)^2;r;x\right)+A_n\left(t(t-x)^2;r;x\right)=$

$A_n\left((t-x)^3;r;x\right)+(1+x)A_n\left((t-x)^2;r;x\right)$

which by (4) and (12) yield (23) for $p,n\in N$.

Let $p\ge 2$. Applying Lemma 2, we get

$$w_p(x)A_n\left(t^p\left((t-x)^2\right);r;x\right)=w_p(x)\left\{\left(x+\frac{1}{a_n}\right)^{p+2}\sum_{j=1}^{p+2}\frac{1}{(a_nx+1)^{2(j-1)}}\right.$$

$$\times\left(\alpha_{p+2,j}+\frac{\gamma_j}{(a_nx+1)^2}+\frac{\beta_{p+2,j}(r)}{(a_nx+1)^2(r-1)!g_{n,r}(x)}\right)-2x\left(x+\frac{1}{a_n}\right)^{p+1}\sum_{j=1}^{p+1}\frac{1}{(a_nx+1)^{2(j-1)}}$$

$$\times\left(\alpha_{p+1,j}+\frac{\gamma_j}{(a_nx+1)^2}+\frac{\beta_{p+1,j}(r)}{(a_nx+1)^2(r-1)!g_{n,r}(x)}\right)+x^2\left(x+\frac{1}{a_n}\right)^p$$

$$\left.\times\sum_{j=1}^{p}\frac{1}{(a_nx+1)^{2(j-1)}}\left(\alpha_{p,j}+\frac{\gamma_j}{(a_nx+1)^2}+\frac{\beta_{p,j}(r)}{(a_nx+1)^2(r-1)!g_{n,r}(x)}\right)\right\}$$

$$=w_p(x)\left\{\left(x+\frac{1}{n}\right)^p\frac{(a_nx+1)^2}{n^2}\left[1+\frac{1}{(nx+1)^2}+\frac{1}{(nx+1)^2(r-1)!g_{n,r}(x)}\right]\right.$$

$$+\sum_{j=2}^{p+2}\frac{1}{(nx+1)^{2(j-1)}}\left(\alpha_{p+2,j}+\frac{\gamma_j}{(nx+1)^2}+\frac{\beta_{p+2,j}(r)}{(nx+1)^2(r-1)!g_{n,r}(x)}\right)$$

$$-2x\left(x+\frac{1}{n}\right)^p\left[\frac{(nx+1)}{n}\left(1+\frac{1}{(nx+1)^2}+\frac{1}{(nx+1)^2(r-1)!g_{n,r}(x)}\right)\right]$$

$$+x^2\left(x+\frac{1}{n}\right)^p\left(1+\frac{1}{(nx+1)^2}+\frac{1}{(nx+1)^2(r-1)!g_{n,r}(x)}\right)$$

$$\left.+\sum_{j=2}^{p}\frac{1}{(nx+1)^{2(j-1)}}\left(\alpha_{p,j}+\frac{\gamma_j}{(nx+1)^2}+\frac{\beta_{p,j}(r)}{(nx+1)^2(r-1)!g_{n,r}(x)}\right)\right\},$$

which by (11) and (4) imply

$$w_p(x)A_n\left(t^p\left((t-x)^2\right);r;x\right)$$

$$\leq w_p(x)\left(x+\frac{1}{a_n}\right)^p\frac{1}{a_n^2\left(1+x^p\right)}\left\{2+r+\sum_{j=2}^{p+2}\frac{1}{(a_nx+1)^{2(j-1)}}\right.$$

$$\times\left(\alpha_{p+2,j}+\gamma_{j+}r\beta_{p+2,j}(r)\right)-2x\sum_{j=2}^{p+1}\frac{1}{(a_nx+1)^{2(j-1)}}\left(\alpha_{p+1,j}+\gamma_j+r\beta_{p+1,j}(r)\right)$$

$$\left.+x^2\sum_{j=2}^{p}\frac{1}{(a_nx+1)^{2(j-1)}}\alpha_{p,j}+\gamma_j+r\beta_{p,j}(r)\right\}$$

$$\leq\frac{M_4}{a_n^2}$$

for $x\in R_0$ and $n,r\in N$.

4. Approximation Behaviour of A_n

In this section, we will investigate the approximation behaviour of A_n.

Theorem 1. Let $p \in N_0$ and $r \in N$ be fixed numbers. Then there exists a positive constant $M_5 = M_5(p,r)$ depending only on the parameters p and r such that for every $f \in C_p^1$ and $r \in R_0$ we have

$$\left\| A_n(f;r;.) - f(.) \right\|_p \le \frac{M_5}{a_n} \left\| f' \right\|_p, n \in N. \quad (24)$$

Proof. Let $x \in R_0$ be a fixed point. Then for $f \in C_p^1$ and $t \in R_0, t \ge x$ we have

$$f(t) - f(x) = \int_x^t f'(u)\, du. \quad (25)$$

By linearity of A_n, (24) and (5) we have

$$A_n(f(t);r;x) - f(x)) = A_n\left(\int_x^t f'(u)\, du; r; x \right), n \in N. \quad (26)$$

From (3) and (4) we obtain that

$$\left| \int_x^t f'(u)\, du \right| \le \left\| f' \right\|_p \left[\frac{1}{w_p(x)} + \frac{1}{w_p(t)} \right] |t-x|, \quad t, x \in R_0. \quad (27)$$

Then, we get

$$w_p(x) \left| A_n(f(t);r;x) - f(x) \right| = w_p(x) \left| A_n \int_x^t f'(u)\, du \right| \le \left\| f' \right\|_p \left\{ A_n(|t-x|;r;x) + A_n\left(\frac{|t-x|}{w_p(x)}; r; x \right) \right\} \quad (28)$$

for $n \in N$. By the Hölders inequality and by Lemmas 1-4 and by (12) it follows that

$$A_n(|t-x|;r;x) \le \left\{ A_n\left((t-x)^2;r;x\right) \right\}^{\frac{1}{2}} \left\{ A_n\left(\frac{1}{w_p(t)}; r; x \right) \right\}^{\frac{1}{2}} \le \frac{M_6(p,r)}{a_n}$$

$$w_p(x) A_n\left(\frac{|t-x|}{w_p(x)}; r; x \right) \le w_p(x) \left\{ A_n\left(\frac{(t-x)^2}{w_p(t)}; r; x \right) \right\}^{\frac{1}{2}} \left\{ A_n\left(\frac{1}{w_p(t)}; r; x \right) \right\}^{\frac{1}{2}} \le \frac{M_7(p,r)}{a_n}, n \in N. \quad (29)$$

Hence and by (28) and (29) we obtain (24).

5. Rates of Convergence

In this section, we compute the rate of convergence of $A_n(f;r;.)$ to $f(.)$ by means of the weighted modulus of continuity given by (6).

Theorem 2. Let $p \in N_0$ and $r \in N$ be fixed numbers. Then there exists a positive constant $M_8 = M_8(p,r)$ depending only on the parameters p and r such that for every $f \in C_p^1$ and $n \in N$ we have

$$\left\| A_n(f;r;.) - f(.) \right\|_p \le M_8 w_1\left(f; C_p; \frac{1}{a_n} \right), n \in N. \quad (30)$$

Proof. Let $f \in C_p$. We use the Steklov function

$$f_h(x) := \frac{1}{h} \int_x^t f(x+u)\, du, x \in R_0, h > 0. \quad (31)$$

From (31) we can write

$$f_h'(x) = \frac{1}{h} \Delta_t f(x), x \in R_0, h > 0 \quad (32)$$

which imply

$$\|f_h - f\|_p \le w_1(f; C_p; h), \tag{33}$$

$$\|f_h'\|_p \le h^{-1} w_1(f; C_p; h) \tag{34}$$

for $h > 0$. From this we deduce that $f_h \in C_p^1$ if $f \in C_p$ and $h > 0$. Hence for (32) we can write

$$w_p(x)(A_n(f; r.x) - f(x)) \le$$
$$w_p(x)\{|A_n(f - f_h; x)| + |A_n(f_h; x) - f_h(x)|$$
$$+ |f_h(x) - f(x)|\}$$
$$:= L_1(x) + L_2(x) + L_3(x)$$

for $n \in N, h > 0$ and $x \in R_0$. For $L_1(x)$, by using Lemma 3 and (33), we get

$$\|L_1\|_p \le M_1 \|f - f_h\| \le$$
$$M_1 w_1(f; C_p; h), \|L_3\|_p \le w_1(f; C_p; h).$$

Similarly, by using Theorem 1 and (34) it follows that

$$\|L_2\|_p \le \frac{M_2}{a_n}\|f_h'\|_p \le \frac{M_2}{a_n h} w_1(f; C_p; h)\ h > 0, n \in N.$$

Consequently,

$$\|A_n(f; r; .) - f(.)\|_p \le$$
$$\left(1 + M_1 + \frac{1}{a_n h} M_2\right) w_1(f; C_p; h).$$

Now, for fixed $n \in N$, setting $h = \frac{1}{a_n}$ in the last equation we obtain

$$\|A_n(f; r; .) - f(.)\|_p \le M_8(p, r)\, w_1\left(f; C_p; \frac{1}{a_n}\right).$$

From Theorems 1 and 2 we will give the followings corollaries:

Corollary 1. For every fixed $r \in N$ and $f \in C_p, p \in N_0$ we have

$$\lim_{n\to\infty} \|A_n(f; r; .) - f(.)\|_p = 0. \tag{35}$$

Corollary 2. For every fixed $r \in N$ and $f \in C_p^1, p \in N_0$ then

$$\|A_n(f; r; .) - f(.)\|_p = o\left(\frac{1}{a_n}\right) \tag{36}$$

as $n \to \infty$.

Theorem 1 and Corollaries in our paper show that the operator A_n, $n \in N$ give better degree of approximation of functions $f \in C_p, f \in C_p^1$ than classical Szász-Kantorovich operators. Because degree of our operators convergence is $\frac{1}{a_n}$ but classical Szász- Kantorovich operators's degree of convergence is $\frac{1}{n}$.

Theorem 3. Let $r \in N$ be fixed number. Then for all $f \in C_p^1$ and $r \in N$ we have

$$\lim_{n\to\infty} a_n \{A_n(f; r; x) - f(x)\} = f'(x) \tag{37}$$

for every $x > 0$.

Proof. Let $x > 0$ be a fixed point. Then by Taylor Formula we get

$$f(t) = f(x) + f'(x)(t - x) + \varepsilon(t; x)(t - x)$$

for $t \in R_0$ where $\varepsilon(t) \equiv \varepsilon(t; x)$ is a function belonging to C_p and $\varepsilon(x) = 0$. Hence by (11) and (13)-(16) we have

$$A_n(f; r; x) = f(x) +$$
$$f'(x) A_n((t - x); r; x) + A_n(\varepsilon(t)(t - x); r; x). \tag{38}$$

By the Hölders inequality and (38) we have

$$A_n(\varepsilon(t; x)(t - x); r; x) \le$$
$$\{A_n(\varepsilon^2(t; x); r; x)\}^{\frac{1}{2}} \{A_n((t - x)^2; r; x)\}^{\frac{1}{2}}.$$

By Corollary 1 we deduce that

$$\lim_{n\to\infty} A_n(\varepsilon^2(t); r; x) = \varepsilon^2(x) = 0.$$

From above equation and Lemma 1 we get

$$\lim_{n\to\infty} a_n A_n\left(\varepsilon(t)(t-x);r;x\right)=0.$$

Theorem 2 show that rate of our operators for pointwise convergence is more fast than classical Szász-Kantorovich operators. Because our operators rate of convergence is $\frac{1}{a_n}$ but classical Szász-Kantorovich operators rate of convergence is $\frac{1}{n}$ pointwisely.

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