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Nonlinear Free Vibration of a Cantilever Beam Using the Power Series Method

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Abstract: An analytical approach based on the power series method is used to analyze the free vibration of a cantilever beam with geometric and inertia nonlinearities. The time variable is transformed into a "harmonically oscillating time" variable which transforms the governing equation into a form well-conditioned for a power series analysis. Rayleigh's energy principle is also used to determine the vibration frequency. Convergence of the power series solution is demonstrated and excellent agreement is seen for the vibration response with a numerical solution.

Key words: Nonlinear vibration, cantilever beam, power series method.

1. Introduction

The analysis of nonlinear vibration systems has traditionally relied on the use of the LP (Lindstedt-Poincare) perturbation [1-3] and the HB (harmonic balance) [4-6] methods. However, the use of the LP perturbation method is often limited to cases where the nonlinear parameters are small thus precluding the treatment of relatively large oscillations. The HB method can be used to analyze large amplitude oscillations but the computational labor required becomes overwhelming.

Recently, several techniques have been developed to treat oscillations with strong nonlinearities, such as the modified LP method [7], the power series method [8] and the homotopy analysis method [9].

In this paper, the free vibration of a conservative cantilever beam with geometric and inertia nonlinearities is obtained using the power series method. This system has been analyzed by Joubari et al. [10] using the modified iteration perturbation method which combines Mickens and iteration methods. The accuracy of the power series solution is checked by a comparison of the vibration frequency obtained by the two methods for different values of vibration amplitude and nonlinear parameters. The displacement and velocity responses obtained by the present method are also compared with a numerical solution.

2. Analytical Formulation

Consider the free vibration of a cantilever beam with nonlinearity in the stiffness and the inertia as governed by the differential equation:

$$\ddot{u}+\alpha\,(u^2\ddot{u}+u\,\dot{u}^2)+u+\beta\,u^3=0 \qquad (1)$$

Subject to the initial conditions: $u(0)$ = A and $\dot{u}(0)$ = 0. The solution of this conservative oscillator is periodic. In order to facilitate the use of the power series method to capture periodic motion, an oscillating time variable [8] is introduced as:

$$\tau=\sin(\omega t) \qquad (2)$$

which starts at $\tau=0$ when t = 0 and oscillates between the values -1 and +1 at a frequency ω as t is increased indefinitely. The infinite time scale $0\le t\le\infty$ is thereby reduced to a finite time domain $-1\le\tau\le1$. When Eq. (1) is transformed from the u-t plane to the u-τ plane in accordance with Eq. (2), the transformed differential equation and initial conditions become:

$$\omega^2[(1-\tau^2)u''-\tau u'](1+\alpha u^2)+\alpha\omega^2(1-\tau^2)uu'^2+u+\beta\,u^3=0 \qquad (3)$$

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$$u(0) = A \text{ and } u'(0) = \dot{u}(0)/\omega = 0$$

where the prime denotes differentiation with respect to τ. The frequency ω is as yet undetermined. Eq. (3) has an ordinary point at $\tau = 0$ and two regular singular points at $\tau = \pm 1$. For linear vibration ($\alpha = 0$, $\beta = 0$), differential equation theory guarantees convergent power series expansion about $\tau = 0$ with a radius of convergence $|\tau| < 1$. This convergence interval covers the infinite time domain except at the singular points. However, for nonlinear differential equations, the question of convergence is still not settled. Here, it is assumed that a convergent power series expansion about $\tau = 0$ exists. For $|\tau| < 1$, as:

$$u(\tau) = a_1 + a_2\,\tau + a_3\tau^2 + \cdots = \sum_{n=1}^{\infty} a_n\tau^{n-1} \quad (4)$$

where a_i is constant coefficients to be determined. Using Eq. (4), the remaining terms in Eq. (3) can be expanded as:

$$u^2 = \sum_{n=1}^{\infty} b_n\tau^{n-1} \quad (5)$$

$$u^3 = \sum_{n=1}^{\infty} c_n\tau^{n-1} \quad (6)$$

$$\omega^2[(1-\tau^2)u'' - \tau u'] = \omega^2 \sum_{n=1}^{\infty} [n(n+1)\,a_{n+2} - (n-1)^2 a_n]\tau^{n-1} = \sum_{n=1}^{\infty} q_n\tau^{n-1} \quad (7)$$

$$(1+\alpha u^2) = (1+\alpha b_1) + \alpha b_2\tau + \alpha b_3\tau^2 + \cdots = \sum_{n=1}^{\infty} B_n\tau^{n-1} \quad (8)$$

$$(1-\tau^2)uu'^2 = \sum_{n=1}^{\infty} f_n\tau^{n-1} \quad (9)$$

The first term in the transformed Eq. (3) involving the product of two power series can be simplified as follows:

$$\omega^2[(1-\tau^2)u'' - \tau u'](1+\alpha u^2) = \Big(\sum_{n=1}^{\infty} q_n\tau^{n-1}\Big)\Big(\sum_{n=1}^{\infty} B_n\tau^{n-1}\Big) = \sum_{n=1}^{\infty} P_n\tau^{n-1} \quad (10)$$

Upon substituting Eqs. (4), (6), (9) and (10) into Eq. (3), the resulting governing equation is:

$$\sum_{n=1}^{\infty} [P_n + \alpha\omega^2 f_n + a_n + \beta c_n]\tau^{n-1} = 0 \quad (11)$$

This equation is satisfied if all the coefficients vanish, which yields:

$$P_n = -a_n - \beta c_n - \alpha\omega^2 f_n\,, n = 1, 2, \ldots.. \quad (12)$$

Imposing the initial conditions results in $a_1 = A$, $a_2 = 0$ giving $b_1 = A^2$, $b_2 = 0$, $c_1 = A^3$, $c_2 = 0$, $f_1 = 0$. To facilitate programming the recursive relation, the first expression (n = 1) in Eq. (12) is considered separately and using Eq. (10) becomes as follows:

$$P_1 = q_1 B_1 = \omega^2(2a_3 - 0)B_1 = -A - \beta A^3 - 0$$

So that:

$$a_3 = \frac{-A - \beta A^3}{2\omega^2(1+\alpha A^2)} \quad (13)$$

Similarly, for n > 1, Eq. (12) can, in conjunction with Eq. (10), be written as:

$$\omega^2[n(n+1)a_{n+2} - (n-1)^2 a_n]B_1 + \sum_{k=1}^{n-1} q_k B_{n-k+1} = -a_n - \beta c_n - \alpha\omega^2 f_n$$

Giving the recursive equation:

$$a_{n+2} = \frac{(n-1)^2}{n(n+1)} a_n - \frac{a_n + \beta c_n + \alpha\omega^2 f_n + \sum_{k=1}^{n-1} q_k B_{n-k+1}}{n(n+1)\omega^2(1+\alpha A^2)} \quad (14)$$

n = 2, 3, …

As might be expected, all the coefficients a_3 and higher can be written in terms of a_1 and a_2 which are the two fundamental coefficients connected with the initial conditions. In addition, all these coefficients depend on the oscillating time frequency ω. The computation of the oscillating time frequency is made possible by noting that, for a conservative system, the sum of kinetic and potential energies is constant. For a given set of initial conditions, periodic motion is represented in the phase plane by perpetual movement around a closed orbit of constant energy, completing one orbit in one period of vibration. Since the point of initial conditions on the orbit is defined uniquely by $\tau = 0$, the oscillating time must circle one orbit in one-half cycle of vibration. As a consequence, the frequency of oscillating time ω is one-half of the vibration frequency Ω:

$$\omega = \frac{\Omega}{2} \tag{15}$$

For the system under consideration, the kinetic and potential energies *T* and *U* respectively are given by:

$$T = \frac{1}{2}(1 + \alpha u^2)\dot{u}^2, U = \frac{1}{2}u^2 + \frac{1}{4}\beta u^4 \tag{16}$$

The motion is assumed to start at $\tau = 0$ with maximum displacement so that the equilibrium position associated with maximum velocity is reached at angular positions $\Omega t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ for which $\omega t = \frac{\pi}{4}, \frac{3\pi}{4}, \ldots \ldots$ and $\tau = \pm\frac{1}{\sqrt{2}}$.

The velocity in the *x*-*t* plane is related to that of the x-τ plane as:

$$\dot{u} = \pm\,\omega\sqrt{(1-\tau^2)}u' \tag{17}$$

Rayleigh's energy principle for conservative systems which equates the maximum potential and kinetic energies may now be used to determine the oscillating time frequency. Now $U_{max} = \frac{1}{2}A^2 + \frac{1}{4}\beta A^4$ and the maximum kinetic energy:

$$T_{max} = \frac{1}{2}(1 + \alpha u^2)(1-\tau^2)\omega^2 u'^2$$

is evaluated at $\tau = \frac{1}{\sqrt{2}}$, corresponding to maximum velocity. The oscillating time frequency ω is a root of the function:

$$w = U_{max} - T_{max} = 0 \tag{18}$$

Another root may exist at a change of sign of the function but the correct root always makes the function w a stationary minimum. The vibration frequency Ω is then twice that value and the corresponding series coefficients a_i determine uniquely the periodic motion, which can be written directly in terms of time *t* as:

$$u(t) = a_1 + a_3 sin^2\omega t + a_5 sin^4 \omega t + \cdots \tag{19}$$

where all the even-numbered series coefficients vanish as a result of imposing the condition of zero initial velocity.

3. Numerical Illustration

The free vibration of the cantilever beam, Eq. (19), was computed for the initial conditions: $\boldsymbol{u(0) = A}$, **and** $\boldsymbol{\dot{u}(0) = 0}$, where *A* is the amplitude of vibration. As an illustration, the values of the nonlinear parameters taken were $\boldsymbol{\alpha = 0.1, \beta = 10}$.

Table 1 gives a comparison of the vibration frequency Ω for different vibration amplitudes between the present method and the modified iteration perturbation method [10] which predicts the vibration frequency as follows:

$$\boldsymbol{\Omega^2 = \frac{(4 + 3\beta A^2)}{(4 + 2\alpha A^2)}} \tag{20}$$

The number of series terms used to obtain the solution was forty. Good agreement is seen between the two solutions.

The first ten non-zero coefficients (odd-numbered) for this case are shown in Table 2.

Table 1 Vibration frequency Ω for α = 0.1, β = 10.

Vibration amplitude	Perturbation [10]	Power series
1	2.8452	2.8002
1.2	3.3177	3.2724
1.6	4.2318	4.1842
1.8	4.6661	4.5824
2	5.0827	5.0230

Figs. 2 and 3 show a comparison of the displacement and velocity time histories, respectively, between the present method and those computed numerically using a fourth order Runge-Kutta scheme. Only the response of one cycle is shown as it is repeated over other cycles. Excellent agreement is seen between the two solutions.

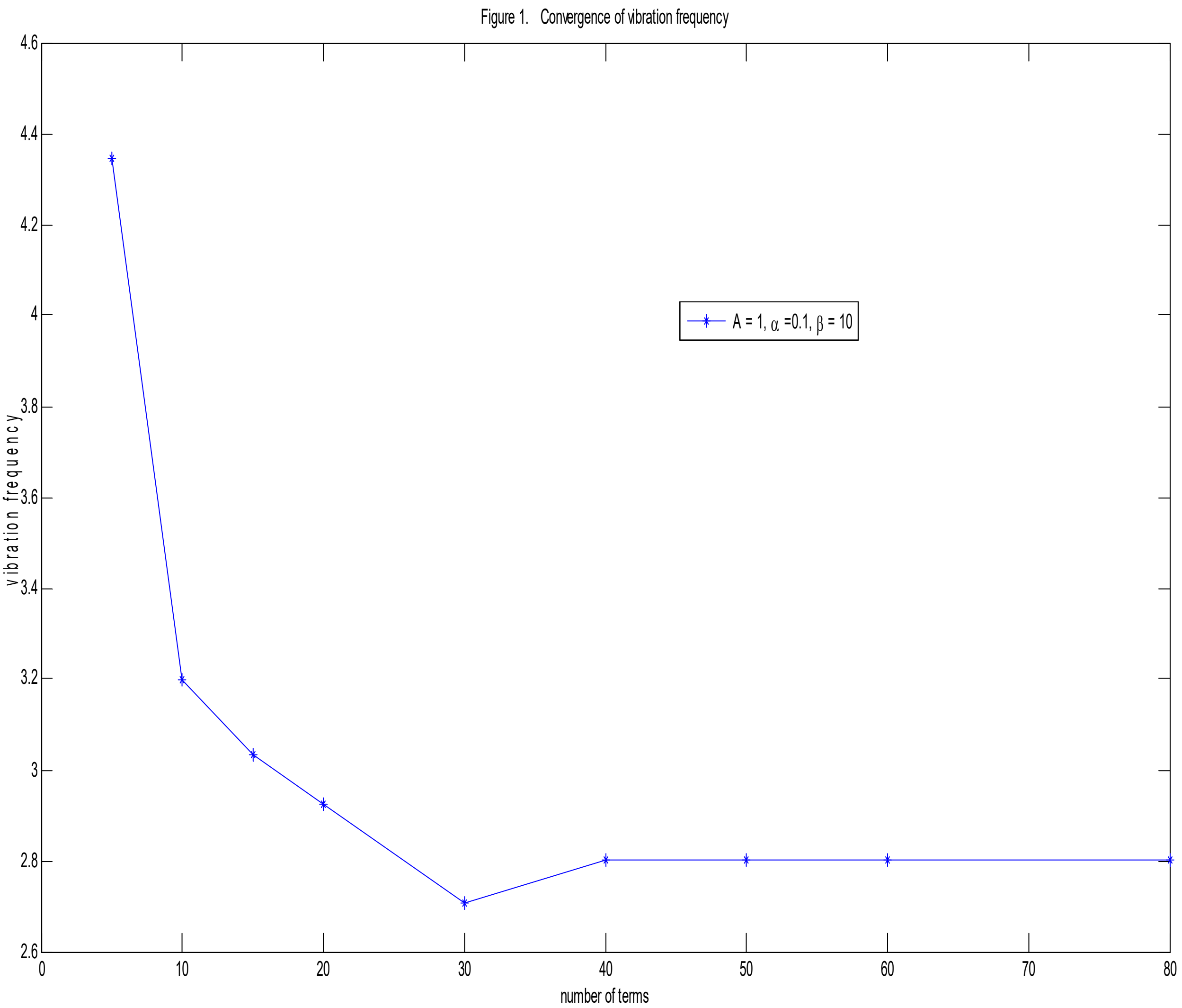

Fig. 1 The convergence of the vibration frequency predicted by the present method as the number of series terms is increased for the case when $A = 1$, $\alpha = 0.1$ and $\beta = 10$.

Table 2 Series coefficients ($A = 1$, $\alpha = 0.1$, $\beta = 10$).

$a_1 = 1.0000$	$a_3 = -2.5507$	$a_5 = 1.8113$	$a_7 = -1.7648$	$a_9 = 0.8081$
$a_{11} = -0.1756$	$a_{13} = -0.3749$	$a_{15} = 0.4423$	$a_{17} = -0.1959$	$a_{19} = -0.1815$

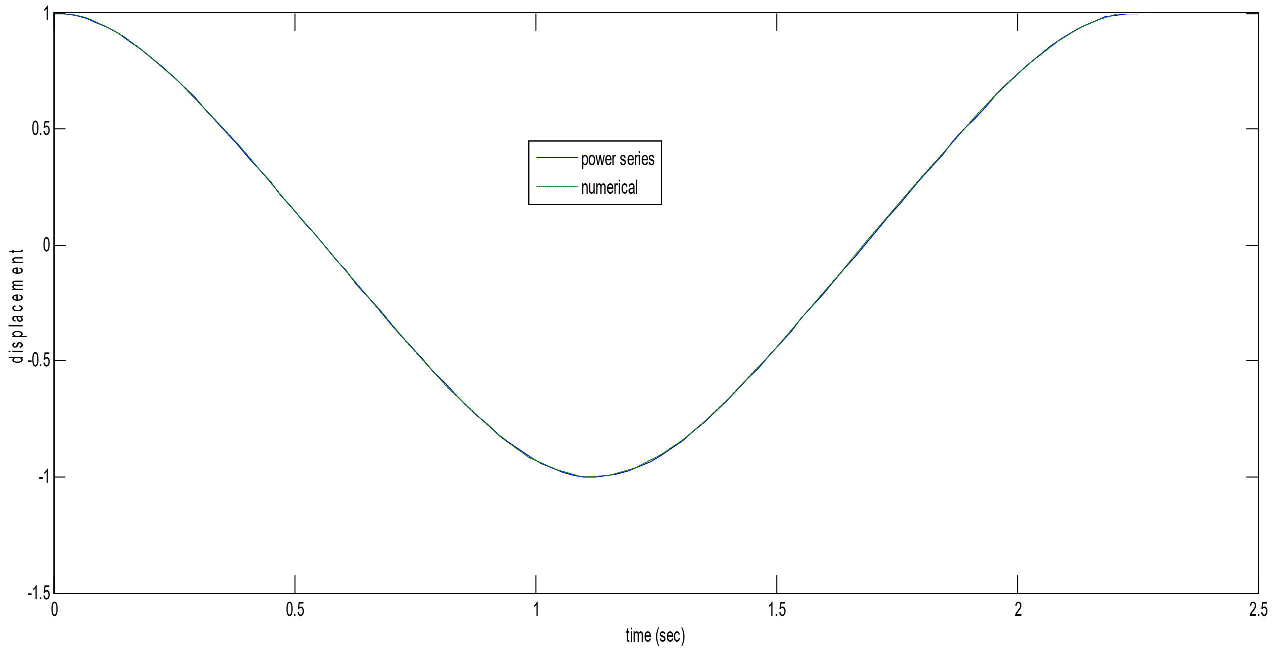

Fig. 2 Displacement response, $\alpha = 0.1$, $\beta = 10$, $A = 1$.

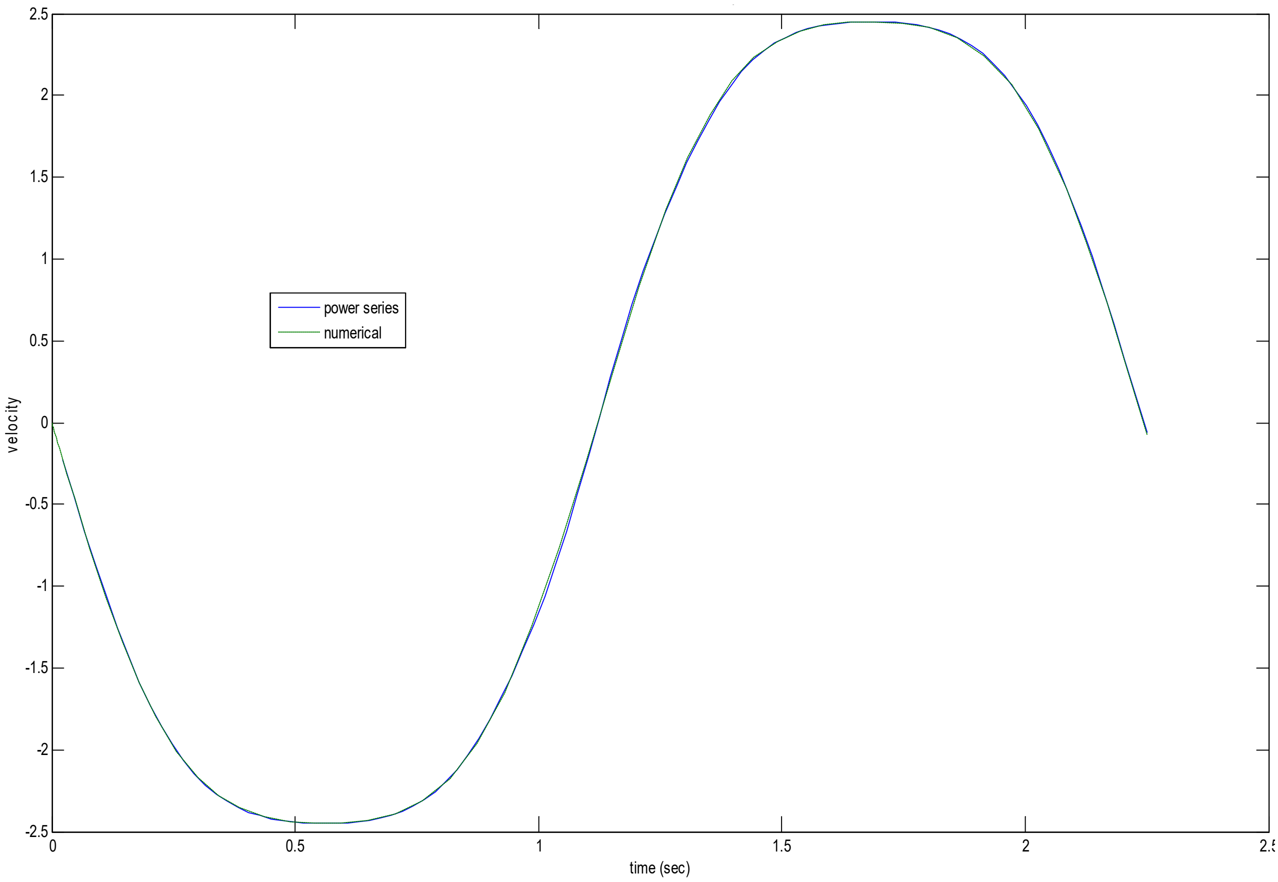

Fig. 3 Velocity response, $\alpha = 0.1$, $\beta = 10$, $A = 1$.

4. Conclusion

A power series solution has been presented to the free vibration of a cantilever beam with geometric and inertia nonlinearities. It is shown that the periodic motion of this conservative system can be represented by a power series expansion convergent for all time when the time variable is transformed by an "oscillating time" variable. The vibration frequency is determined by using Rayleigh's energy principle and the resulting expansion uniquely defines the beam motion.

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