

Continuous-Time Models for Firm Valuation and Their Collateral Effect on Risk-Neutral Probabilities and No-Arbitraging Principle*

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Extensions of Merton's model (EMM) considering the firm's payments and generating new types of firm value distribution are suggested. In the open log-value/time space, these distributions evolve from initially normal to negatively skewed ones, and their means are concave-down functions of time. When payments are set to zero or proportional to the firm value, EMM turns into the Geometric Brownian model (GBM). We show that risk-neutral probabilities (RNPs) and the no-arbitraging principle (NAP) follow from GBM. When firm's payments are considered, RNPs and NAP hold for the entire market for short times only, but for long-term investments, RNPs and NAP just temporarily hold for individual stocks as far as mean year returns of the firms issuing those stocks remain constant, and fail when the mean year returns decline. The developed method is applied to firm valuation to derive continuous-time equations for the firm present value and project NPV.

Keywords: firm present value, geometric Brownian (Structural) model, risk neutral probabilities, no-arbitrage pricing principle

Introduction

Valuation of assets generating long-term cash flows is one of the central problems of financial economics. This valuation uses two main ideas: the time value of money and a rational choice of the expected rate of returns corresponding to asset risks. Among such assets, one can find business projects, firms, bonds, stocks, etc. At that, the cash flow of returns is considered as a stochastic process, characterized by its volatility, but all payments are assumed to be regular functions of time. For example, the project net present value calculated as a discrete process (e.g. Brealey & Myers, 1996, p. 35) is

$$NPV = FCF_0 + \sum_{t=1} \frac{FCF_t}{(1+r_t)^t} \quad (1)$$

$$FCF_t = \langle Return_t \rangle - P_t \quad (1a)$$

here FCF_t , $t = 1, 2, 3 \dots$ is a free cash flow, $\langle Return_t \rangle$ is a dollar value of expected returns, P_t —payment, all at year t . $FCF_0 < 0$ is an initial investment in the project (a down payment), r_t is the expected rate of returns at year t corresponding to the cash flow volatility. The term $FCF_t(1+r_t)^{-t}$ recalculates the free cash flow FCF_t at year t to its value at $t = 0$. When the mean returns, payments, and expected rate of returns are constant:

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$\langle \text{Return}_t \rangle = \langle \text{Return} \rangle$, $P_t = P$, $r_t = r$, then Equation (1) turns into

$$NPV = FCF_0 + \sum_{t=1}^{\infty} \frac{FCF}{(1+r)^t}$$

However, a fixed free cash flow FCF means that the project value grows linearly. The natural economic growth is exponential; so, a better equation is

$$FCF_t = FCF_{t-1}(1+\alpha) = FCF_1(1+\alpha)^t.$$

This FCF_t value with constant α and r ($\alpha < r$) leads to another equation for NPV

$$NPV = FCF_0 + FCF_1 \sum_{t=1}^{\infty} \left(\frac{1+\alpha}{1+r} \right)^t = FCF_0 + FCF_1 \frac{1+r}{r-\alpha} \quad (2)$$

NPV estimate (2) is consistent with a continuous-time geometric Brownian model (GBM) for the project development:

$$\frac{dX_t}{X_t} = \alpha dt + C dW_t, X(0) = X_0, \quad (3)$$

here X_t is a random project value at year t , α —the expected rate of returns, C is the X_t -volatility, and W_t is a Wiener process; α and C are constants. A variable $z = \ln(X_t/X_0)$ follows a standard diffusion process starting at point $z = 0$:

$$dz_t = R dt + C dW_t, z(0) = 0, R = \alpha - C^2/2. \quad (4)$$

The process z_t has a normal distribution $N(z; \langle z_t \rangle, \text{var}(t))$ with its mean and variance

$$\begin{aligned} \langle z_t \rangle &= Rt, \text{var}(t) = C^2 t \\ FCF_t &= \langle X_t \rangle - \langle X_{t-1} \rangle \approx \langle X_{t-1} \rangle (1+\alpha), \\ \langle X_t \rangle &= X_0 \exp [\langle z_t \rangle + \text{var}_t/2] = X_0 \exp(\alpha t). \end{aligned}$$

Equation (2) and its logarithmic equivalent (3) include no payment, while Equation (1a) shows that the payments are allegedly considered. This conflict indicates an internal inconsistency of Equation (2) to be resolved; we shall do it in Section 1. For the diffusion processes (3) or (4), one has an identity used in Equation (1a)

$$\langle \text{Returns}_t - P_t \rangle = \langle \text{Returns}_t \rangle - P_t$$

Later we show that when one takes systematically into account the firm's payments, this equality does not hold and Equation (2) is wrong. It is impossible to improve it remaining within the GBM frames. Equation (2) also supposes an infinite project's time horizon, which is absurd. What factors determine the time horizon of the project, and how long it is, these questions make another objective of our study. The last introductory remark is about volatility. Traditionally, volatility is denoted as σ . However, later we will consider z_0 not as a point but as a normally distributed ensemble of Brownian particles with the mean $\langle z_0 \rangle = H_0$ and variance $\text{var}(z_0) = \sigma_0^2$:

$$\langle z_t \rangle = H_0 + Rt, \text{var}(t) = \sigma_0^2 + C^2 t.$$

To distinguish between the variance and volatility, we denote volatility as C reserving symbol σ for the standard deviation. The expected rate of returns in Equation (2) depends on the time-invariant volatility $r_t = r(C)$, which is also consistent with Equation (3).

Similar equations describe the firm, bond, and stock values. For the firm value, one has

$$PV(\text{firm}) = PV(\text{free mean cash flow}) = PV(\text{mean revenues} - \text{costs} - \text{investments}).$$

For a free cash flow of the form $FCF_t = \langle X_t \rangle - \langle X_{t-1} \rangle \approx \langle X_{t-1} \rangle(1 + \alpha)$, α and $r(C)$ are fixed, the firm present value is

$$PV = \sum_{t=1}^{\infty} \frac{FCF_t}{(1+r)^t} = \sum_{t=1}^{\infty} \frac{FCF_1(1+\alpha)^t}{(1+r)^t} \quad (5)$$

One can see that the firm PV faces the same problems as the NPV estimate for the project (2). Equation (5) claims that it takes account of payments, but GBM (3) consistent with (5) does not include any payment at all; it indicates an internal inconsistency of Equation (5).

The bond value is defined as a sum of the discounted cash flow of coupon payments at a time horizon of the bond maturity plus a discounted face value

$$V_B = \sum_{t=1}^N \frac{INT}{(1+k_d)^t} + \frac{M}{(1+k_d)^N} \quad (6)$$

where M is a bond face value, N is an interval of the bond maturity, INT is a constant nominal value of the coupon, $k_d = k_d(C)$ is the expected rate of returns corresponding to volatility C .

The stock value is estimated as a discounted flow of dividends

$$V_s = PV(\text{expected dividend flow}) = \sum_{t=1}^{\infty} \frac{DIV}{(1+k_s)^t} \quad (7)$$

here DIV is a nominal dividend value, $k_s = k_s(C)$, is the expected rate of returns corresponding to the volatility C . It seems that Equations (6) and (7) do not include any payment, and, therefore, they are free of the conflict specific for the project NPV and the firm PV . However, remembering that the stock value is just a known part of the firm value, and the bond value is a part of the firm's debt, we conclude that Equations (6) and (7) are closely related to Equation (5), and they face the same problems. Equations (6) and (7) must be corrected to

$$V_B = \sum_{t=1}^N \frac{INT(1 - PRD(t))}{(1+k_d)^t} + \frac{M(1 - PRD(N))}{(1+k_d)^N} \quad (6a)$$

$$V_s = \sum_{t=1}^{\infty} \frac{DIV(1 - PRD(t))}{(1+k_s)^t} \quad (7a)$$

where $PRD(t)$ is the firm's default probability at time t , $INT(1 - PRD(t))$ is the expected coupon value and $DIV(1 - PRD(t))$ is an effective dividend value. As in Equations (2) and (5), there is a question about the time horizon in Equation (7). The time horizon in Equation (7a) is governed by the default probability $PRD(t)$. In other words, one must find the firm value distribution, and then the default probability as a function of time and parameters of the firm and its environment. So, to answer the questions raised in the paper, one has to consider the firm valuation problem. There is a vast literature on estimations of the bond and stock values, we discuss some articles in our literature review, but we do not consider details of the valuation of bonds or stocks in our study.

Since the 1970s, the continuous-time diffusion models introduced by Black and Scholes (1973) and Merton (1974) estimate the values of options, firms, bonds, stocks, etc. Black and Scholes, in their classic option pricing study, use GBM (3) that leads to the lognormal distribution of option prices. Merton (1974), estimating the price of a zero-coupon bond of a given maturity issued by the firm that defaults when the asset value of the firm is less than the outstanding debt at the time of debt maturity, introduces an equation taking account of debt and dividend payments. However, instead of solving this equation, he puts forward the famous Option hypothesis saying "While options are highly specialized and relatively unimportant financial instruments [...] the same basic approach could be applied in developing a pricing theory for corporate

liabilities in general” (Merton, 1974, p. 449). This hypothesis allows him to be satisfied with the GBM-solution which is, of course, identical to the Black-Scholes solution. Merton has interpreted this identity as a proof of the general validity of his option hypothesis.

Now the Option hypothesis dominates worldwide in the financial community. For example, Strebulaev and Whited (2012, pp. 4-5), reviewing the development of dynamic structural models, state that “they (dynamic structural models) start with the acknowledgment that any claims on corporate cash flow streams are derivatives on underlying firm value or firm cash flows. This means that we can apply option pricing methods to value these claims.” Black and Cox (1976) improve the Merton model introducing a threshold triggering default when the firm’s assets hit the threshold. Since that time, the structural model becomes the most popular instrument of study in different fields of economics and finance. Sundaresan (2013, p. 21) declares that “since its publication, the seminal structural model of default by Merton (1974) has become the workhorse for gaining insights about how firms choose their capital structure, a “bread-and-butter” topic for financial economists.” Numerous GBM modifications are used in plenty of papers studying various aspects of corporate development such as a search for the optimal capital structure (Leland, 1994; Leland & Toft, 1996; Leland, 2006; etc.), dynamic methods of the debt control (Goldstein et al., 2001; Strebulaev, 2007; Titman & Tsyplakov, 2007; Hugonnier et al., 2015; etc.), analysis of the relation between the macroeconomic state of the economy and the intensity and scale of defaults (Chen, 2010; Bhamra et al., 2010; etc.), investigation of the dependence between the firm’s investment decisions and its financing decisions (Barclay & Smith, 1995; Barclay, Smith, & Morellec, 2006; etc.), and many others. Further examples of the application of the structural models one can find in comprehensive reviews of Strebulaev and Whited (2012), Laajimi (2012), and Sundaresan (2013). Because the normal distribution is a solution of the standard diffusion equation about the firm value, and the lognormal one is a solution of the same equation about the log-value, these two distributions have a lot of remarkable properties such as: (a) a time-sequence of the calibrated firm’s stock prices makes a martingale, (b) there exists a risk-neutral (or martingale) measure and risk-neutral probabilities significantly simplifying the analysis of the firm’s credit risks and default, and (c) a market with martingale prices is the no-arbitraging market (Cox et al., 1979; Harrison & Kreps, 1979; Harrison & Pliska, 1981).

However, the default probabilities predicted by GBM occur much lesser than the default frequencies observed in practice; it means that the real log-value distribution has heavy tails, or is negatively skewed. Since the end of the 1980s, theoretical studies on the firm value distribution and default probabilities split into two directions. The jump-diffusion processes (JDPs) supplementing GBM with Poisson jumps in the firm value make the first (e. g. Zhou, 2001; Hilberink & Rogers, 2002; Kou, 2002; Chen & Kou, 2009). The jumps allegedly represent a market reaction to new information about the firm, and dominating leaps down provide for desired negative skewness to the firm value distribution. Estimation of constant jump parameters, their intensity and mean length, makes a specific problem usually resolved with calibrated models (Leland, 2006).

Giesecke and Goldberg (2008) use a structural model of credit risk to show that informational asymmetries can induce an event premium for the abrupt changes in security prices that occur at default. If the public investors are unable to observe the threshold asset value at which the firm’s management liquidates the firm, then they face an abrupt default risk as they cannot discern the firm’s distance to default. Technically the authors suggest another kind of JDP adding an extra jump risk to a low GBM risk. To apply the martingale technique to solving of the problem, Giesecke and Goldberg use GBM supplemented with jumps in the firm value to a default line whose location is random. Random leaps to the default line simulate an unexpected occurrence of the firm’s default.

However, this unexpected default can be explained by the difference between the real high default probability and its low GBM estimate used by investors as well as the firm's management. In this case, both public investors and the firm's management use the same information about the firm and are almost equally unpleasantly surprised by the firm's default. Both types of JDP models supplement GBM with jumps in the firm value, although, as one believes, the leaps have different causes: an appearance of new information about the firm (Zhou, 2001; Hilberink & Rogers, 2002; Kou, 2002; Chen & Kou, 2009), or informational asymmetries between public investors and firm's management (Giesecke & Goldberg, 2008). Both types are equally far from reality because any firm *has* compulsory payments, which JDP models neglect. As we show further, the real log-value distribution is negatively skewed having a growing left tail. No JDPs with fixed statistical jump parameters can provide for the growing left distribution tail.

The second group of theoretical models aimed to achieve default probabilities comparable with the default frequencies observed in practice consists of stochastic volatility processes providing for symmetric distributions with heavy tails (e.g. Hull & White, 1987; Melino & Turnbull, 1990; Nicolato & Venardos, 2003). This group of models is mainly used for option pricing and is not considered here.

So-called calibrated models demonstrate another attempt to take heuristic account of the distribution skewness. We show the shortcomings of this type of model on the example of Moody's KMV (Bohn, 2006). To introduce negative skewness to the log-value distribution, the model uses an extensive database of real defaults for estimating default probabilities and the loss distribution at a time horizon of one year. The model applies to publicly traded firms for whom market values are known. To determine a firm's current state, the model uses GBM to calculate the distance-to-default (DD) as a height of the log-value mean over a default line measured in standard deviations. Then using the database, the model determines a share of firms with that DD who have defaulted within a year. This share has got the name of Expected Default Frequency (EDF) and is a rough estimate of the intensity of default probability (*IPD*, see Section 1) at a distance of one year. Despite its popularity, the model suffers from serious drawbacks typical for calibrated models. First, the assumption that EDF is a function of DD only is far from reality. Two firms having the same DD at some time can have different *IPD* values because the firm value distribution depends on parameters of the firm and its business environment (the debt leverage, interest rate, inflation rate, taxation rate, etc.). Second, for credit risk estimation objectives, a creditor wants to know the probability of borrower's default at a horizon of the credit maturity, which can achieve decades while Moody's KMV works at the time horizon of one year only. The natural conclusion from all said above is we need a more accurate and precise model for the firm value distribution.

In this paper we take an attempt to present two such models (we call them the First and the Second extended Merton models, or EMM1 and EMM2 for short) taking account of the firm's payments and breaking off any connection with GBM following from the false Merton's analogy between the firm and the option. We show that the EMM-firm and the GBM-firm have very different characteristics both at the firm level and at the market level. The GBM-firm remains "ever young", keeping the time-invariant mean year returns and volatility and producing over optimistically low default probabilities. On the contrary, the EMM-firm by and by "grows old": its mean year returns decrease after some time depending on the firm parameters, its volatility and negative skewness continuously grow contributing to the default probability. At the market level, the GBM-firms admit the time-independent risk-neutral measure, risk-neutral probabilities, and no-arbitrage pricing principle effective for the entire market. The market of the EMM-firms does not have a risk-neutral measure independent of time and for the whole market. However, each firm at the market can have its

risk-neutral measure for some time determined by parameters of the firm and its business environment. The implication from this is that the no-arbitraging property becomes a feature of the firm, holding only for some limited time. Risk-neutral probabilities exist as far as the mean year returns of the firm remain constant; thus, the risk-neutral approach is legitimate for safe firms only. One cannot use this approach for estimating credit risks and finding the firm's default probabilities.

The rest of the paper has the following structure. Section 1 presents a continuous-time model estimating the probability that a firm with *continuous payments* will meet in its development financial difficulties that bring about the firm's default. We show that for the firm paying its compulsory expenses, the firm value distribution is negatively skewed, and reveal a dependence of the default probability on parameters of the firm and its business environment. Knowledge of a measure of the firm's stability is necessary to its long-term investors, creditors, and also to the firm's management planning long-term business operations. At the end of the Section, we give the continuous-time equation for the firm present value taking account of the firm's payments.

In Section 2, we present and solve EMM2 for the firm with *discrete payments* when the firm pays its compulsory expenses in a lump sum at the end of each year. The EMM2-firm and the EMM1-firm have very similar general properties at the individual (firm) level as well as at the market level. We show that the risk-neutral probabilities and no-arbitraging property remain invalid for both types of the firm's payments, continuous or discrete. For both EMM1- and EMM2-firms, the risk-neutral default probabilities and no-arbitraging property become time-dependent characteristics of individual stocks and the firms issuing them, rather than the feature of the market as a whole. From a practical point of view, it means that long-term investors such as pension funds, mutual funds, banks, and big firms suffer unnecessary losses under the wrong impression of correctness of the GBM-estimations and the no-arbitraging principle. We show that the validity of the risk-neutral probabilities and no-arbitraging property does not depend on the type of the firm's payments, continuous or discrete. In the end of this Section, we present results of computer modeling of the firm value distribution and its statistical moments for various initial conditions supporting our qualitative analysis (EMM1).

Model Description

In his seminal work, Merton (1974) introduces a continuous-time equation describing the firm value developing in a stochastic environment (the general Merton model):

$$dX = (\alpha_0 X - P)dt + CXdW, \quad X(0) = X_0, \quad (1.1)$$

$$P = DP + DIV \quad (1.1a)$$

here $X(t)$ is the firm market value at time t , constant μ is a rate of instantaneous expected returns on the firm per unit time, P is the total dollar payouts by the firm per unit time to either its shareholders or liabilities-holders (dividend DIV or interest DP payments) per unit time, constant C^2 is the instantaneous variance of returns, W is a Wiener process representing a cumulative effect of normal shocks. (For the sake of consistency with the further discussion, we use our symbols for the variables and parameters in the model keeping the original Merton's interpretation of symbols. One should note, however, that C^2 is not the variance of returns, but the rate of variance growth and C is the process volatility.)

Merton solves a specific case of this equation with $P = 0$. For $P = \delta_0 X$, $\delta_0 \geq 0$ is constant, Equation (1.1) transforms into the geometric Brownian model, GBM:

$$(dX)/X = \lambda dt + CdW, \quad \lambda = \begin{cases} \alpha_0, & P = 0 \\ \alpha_0 - \delta_0, & P = \delta_0 X \end{cases} \quad (2.1)$$

From Equation (2.1) it follows that GBM requires very restrictive conditions for its validity. The condition $P = 0$ means that a firm makes no payments at all (Black & Scholes, 1973; Merton, 1974; Black & Cox, 1976; Leland, 1994; etc.). The condition $P = \delta_0 X$ used in (Leland, 1994; Leland & Toft, 1996; Goldstein et al., 2001; J. Huang & M. Huang, 2012, etc.) is more or less acceptable when payment P consists of dividends only, but when P includes dividends as well as debt payments, this proportionality becomes doubtful. Payment P has its schedule hardly related to changes in the firm's size and value, and they certainly do not follow the firm value when it drops to the default threshold. A GBM solution is a lognormal distribution

$$U(X, t) = (2\pi\sigma^2)^{-1/2} X^{-1} \exp [-(\ln X - H)^2 / (2\sigma^2)] \quad (3.1)$$

$$H(t) = H_0 + Rt, \quad \sigma^2(t) = \sigma_0^2 + C^2 t, \quad R \equiv \lambda - C^2/2$$

In the original Merton model, a firm defaults only if its value is less than the firm's outstanding debt at the time of debt maturity. Black and Cox (1976) improve this shortcoming introducing a threshold triggering default any time when the firm value hits the threshold (a default line). This version of the model spreads widely, and all subsequent generalizations of the model using GBM in their core historically have got the name of structural models. An excellent introduction to modern methods of credit risk estimation one can find in (Crouhy, Galai, & Mark, 2006; The Credit Market Handbook, 2006).

Because of a role that the firm value has in financial economics, the general Merton model and its solution are of great importance. However, before studying this model, we revise it because Merton's interpretation of payments is too short. The revised model considers firm manufacturing and marketing its production or rendering services at a market, subject to random shocks of a normal distribution. The market shocks affect the firm value with intensity C . The firm makes various payments while doing its business. Some of them are due to the manufacturing and marketing of the firm's goods (variable costs); one can take them into account by adjusting the rate of returns α_0 . Other payments secure the every firm's presence in business. Such expenses include fixed costs (FC), taxes (TAX), dividends (DIV), and debt payments (DP), all per unit time. Thus, one can write for *business securing expenses (BSEs)*

$$P = FC + DP + TAX + DIV, \quad (4.1)$$

$$P(t) = P_0 \pi(t), \quad P(0) = P_0 > 0, \quad \pi(0) = 1.$$

here P is an arbitrary *continuous* function of time; P_0 is a positive constant. The time dependence of FC and DP reflects changes in business conditions; TAX and DIV depend on their rates and year returns. Here after we refer to the process (1.1), (4.1) as the First extended Merton model or EMM1 for short.

Equation (1.1) for random variable $x = \ln(RX/P_0)$ by Ito's Lemma transforms into

$$dx = R(1 - \pi(t)e^{-x})dt + CdW, \quad (5.1)$$

$$x(0) = x_0 = \ln(RX_0/P_0), \quad R = \alpha_0 - C^2/2. \quad (5a.1)$$

Writing a Fokker-Plank equation for Equation (5.1), one comes to an equation for the probability distribution $V(x, t)$, or x -distribution; V_y is a partial derivative over a variable y :

$$V_t + R(1 - \pi(t)e^{-x})V_x - 0.5C^2V_{xx} + R\pi(t)e^{-x}V = 0. \quad (6.1)$$

The initial condition is

$$V(x, 0) = V_0(x; H_0, \sigma_0^2), \quad (7.1)$$

$$H_0(t) = \langle x(0) \rangle = \int_{-\infty}^{\infty} xV(x, 0)dx, \sigma_0^2(t) = \langle (x - H_0)^2 \rangle = \int_{-\infty}^{\infty} (x - H_0)^2 V(x, 0)dx,$$

where $V_0(x; H_0, \sigma_0^2)$ is a normal distribution. There is also a boundary condition implying that a firm will default when its value falls to X_D ($X_D < X_0$)

$$V(DL, t) = 0, DL = \ln(RX_D/P_0). \quad (8.1)$$

If X_D is an outstanding debt as it is in (Black & Cox, 1976), Equation (8.1) makes an *exogenous* constraint. If the firm is free of debt, there is another constraint. A BSE share in the expected year returns is

$$P_0/(R\langle X_0 \rangle) = \exp(-H_0 - \sigma_0^2/2). \quad (9.1)$$

For $H_0 \geq 0$, this share is less than unit, while for $H_0 < 0$, it is more than unit, and the firm pays out more than it earns. The line $x = 0$ separates a profitable business from its failure. In this case, it is reasonable to introduce a soft *endogenous* boundary

$$V(DL, t) = 0, DL = 0, \quad (10.1)$$

and watch the probability of crossing this line. Below the line, the firm's activities are possible only if selling some other firm's equity. The nature of this boundary is close to the default line introduced by Kim, Ramaswamy, and Sundaresan (1993). The firm defaults at this line if it runs out of cash. The boundary conditions (8.1) and (10.1) can be joined as

$$V(DL, t) = 0, DL = \max[0, \ln(RX_D/P_0)]. \quad (11.1)$$

A solution of the boundary problem (6.1), (7.1), (11.1) is the firm log-value distribution; it is denoted as $\hat{V}(x, t)$. If one knows $V(x, t)$ solution in the open space, then a solution of the boundary problem can be written as

$$\hat{V}(x, t) = V(x, t) - V(2DL - x, t) \quad (12.1)$$

The probability distribution turns into zero at the default line: $\hat{V}(x, t) = 0$, and the intensity of default probability *IPD* is

$$IPD(t) = 2 \int_{-\infty}^{DL} V(x, t)dx \quad (13.1)$$

The first three moments (the mean, variance, and skewness) are calculated along with the probability distribution $V(x, t)$:

$$H(t) = \int_{-\infty}^{\infty} xV(x, t)dx, VAR(t) = \int_{-\infty}^{\infty} (x - H)^2 V(x, t)dx, S(t) = \int_{-\infty}^{\infty} (x - H)^3 V(x, t)dx \quad (14.1)$$

$S(t)$ proportional to distribution skewness shows development of the distribution asymmetry. The main objective of any credit risk analysis is estimating the default probability over a chosen time interval (e.g. over the debt maturity)

$$PRD(t_s, T) = \int_{t_s}^{t_s+T} IPD(t)dt \quad (15.1)$$

here t_s is the moment when the credit is issued, $t_s + T$ is the moment of debt maturity, and $PRD(t_s, T)$ is the default probability over the credit maturity period T . In this paper, $t_s = 0$.

The mean log-value characteristics of the boundary problem (the mean, variance, and skewness) are:

$$\begin{aligned}\hat{H}(t) &= \int_{DL}^{\infty} x \hat{V}(x, t) dx, \quad \hat{V}ar(t) = \int_{DL}^{\infty} (x - \hat{H}(t))^2 \hat{V}(x, t) dx, \\ \hat{S}(t) &= \int_{DL}^{\infty} (x - \hat{H}(t))^3 \hat{V}(x, t) dx\end{aligned}\quad (16.1)$$

Now we can answer the question raised at the beginning of the paper: what is the present value of the firm. The mean dollar value $\hat{X}(t)$ of the firm in a semi-open space with a default line DL is

$$\hat{X}(t) = (P_0/R) \int_{DL}^{\infty} e^x \hat{V}(x, t) dx \quad (17.1)$$

The process volatility is

$$\hat{C}(t) = \left[\frac{d}{dt} \hat{V}ar(t) \right]^{1/2} \quad (18.1)$$

Using CAPM or any other theory providing for the relation between the expected rate of returns and volatility, one can find the expected rate of return as a function of volatility $r(\hat{C})$. The time-continuous equation for the firm present value PV improving the discrete Equation (7) in the Introduction is

$$PV = (P_0/R) \int_0^{\infty} \frac{d\hat{X}}{dt} e^{-t \ln(1+r(\hat{C}))} dt \quad (19.1)$$

where $\hat{C}(t)$ is the process volatility determined by Equations (16.1) and (18.1). The interval of integration in (19.1), or the firm/project time horizon, depends on two conditions: (a) $d\hat{X}/dt > 0$, and (b) $\hat{C} < C_{cr}$, where C_{cr} is critical volatility unacceptably high for the investors. On the eve of achieving conditions $d\hat{X}/dt \leq 0$ or $\hat{C} \geq C_{cr}$, investors begin massively to sell out the firm's stocks dropping their price and the firm value down. The time horizon H_T is determined by the event that occurs first:

$$H_T = \min\{T_X, T_C; T_X: d\hat{X}/dt = 0 \text{ for } t = T_X; T_C: \hat{C}(T_C) = C_{cr}\}, \quad (20.1)$$

$$PV = (P_0/R) \int_0^{H_T} \frac{d\hat{X}}{dt} e^{-t \ln(1+r(\hat{C}))} dt$$

The project NPV is obviously determined by the equation $NPV = FCF(0) + PV$, where PV is given by Equation (20.1).

What Type Is x -Distribution, and How Does It Depend on the Mode of Payment?

For the firm with constant payments, $\pi(t) \equiv 1$, no debt, $x = \ln(RX/P_0)$, Equation (5.1) is

$$dx = R(1 - e^{-x})dt + CdW, \quad (1.2)$$

here payments $P = P_0$ are paid *continuously*. Another way to pay BSEs is to pay them as a lump sum once a year, or *discretely*. We consider both modes of payment starting with the continuous one. Because stochastic Equation (1.2) with initial condition (5a.1) has no exact solution, we try to understand the process behavior in an *open space* ($-\infty < x < \infty$) qualitatively using the Brownian motion model. Suppose that at $t = 0$, we have an ensemble of Brownian particles whose initial locations X_0 have a normal distribution with mean H_0 and standard deviation σ_0 , and one part of this ensemble is over line $x = 0$, while the other is under it (x -axis shows up). At line $x = 0$, there is a balance between mean year returns $R\langle X_0 \rangle$ and payments P .

It follows from Equation (1.2) that the drift rate depends on x . At $x = 0$, the drift is zero; this line is the line of unstable equilibrium for the process $x(t)$. For $x > 0$, the particles drift up, the faster the higher x (due to

positive repulsion from the line $x = 0$). Parameter R bounds the drift rate from above. For $x < 0$, the particles drift down, the faster the greater $|x|$, with no limit for the drift rate at all. The two repulsion forces decrease a concentration $n(x)$ of particles around line $x = 0$, and the concentration of the particles under the line drops faster and lower than the concentration above the line. Thus, a diffusion force appears proportional to the concentration gradient $-dn/dx$ acting across line $x = 0$ and driving the particles located over the line *against* the repulsing force to line $x = 0$. Below line $x = 0$, the diffusion force fades fast because $-dn/dx$ tends to zero, but the negative repulsion force carries the particles farther down. The decreasing concentration of particles in a thin layer over line $x = 0$ makes the particles in the next layer above the first, so far moving up, to stop and then move downwards. The line $x = EQ(t)$, where particles make this U-turn, floats up and up until all particles in the ensemble find themselves under the line and moving down. For $x > EQ(t)$, the particles move up; for $x < EQ(t)$, they move down; and for $x = EQ(t)$, their drift rate is zero. The diffusion force acting between the lines $x = EQ(t)$ and $x = 0$ transports the particles across line $x = 0$, then the negative repulsion force drives them to negative infinity. The initially normal distribution of Brownian particles by and by turns into a leptokurtic and negatively skewed x -distribution. At that, the higher the ensemble location over line $x = 0$, the more time the ensemble deformation takes, the lesser the distribution skewness for fixed time intervals (not too long). The distribution rise in the early stage of its development can to some extent slow down the diffusion mass transfer across line $x = 0$ impeding the skewness development. Nevertheless, the permanently growing distribution skewness makes the distribution mean *a concave-down function of time* for *every* initial location of the ensemble center H_0 . Vice versa, the closer the mean of the initial ensemble to line $x = 0$, the faster runs the distortion of the normally distributed ensemble. The described ensemble evolution explains the space-time development of the x -distribution.

It is interesting to compare the time dependences of the means $\hat{H}(t)$ and $H(t)$, variances $\hat{V}ar(t)$ and $VAR(t)$, and skewness $\hat{S}(t)$ and $S(t)$ for the boundary problem (6.1)-(8.1) and the problem (6.1)-(7.1) in the open space, correspondingly. The means $\hat{H}(t)$ and $H(t)$ behave the same way, they are both concave-down functions of time: $\hat{H}(t)$ because of losses at the boundary, and $H(t)$ because of its ever-growing negative (left) tail. The main difference between them is in the time-scales of their development; because of losses, $\hat{H}(t)$ achieves its maximum and begins to decline sooner than $H(t)$. The variances $\hat{V}ar(t)$ and $VAR(t)$ behave quite differently. In the first part of its evolution, $\hat{V}ar(t)$ is a concave-up function of time growing due to the diffusion expansion and the distribution deformation. Then its growth slows down to zero with $\hat{V}ar(t)$ achieving its maximum as the distribution's expanding and shrinking tendencies come to a short-time balance. In the last part of its evolution, the variance declines to zero as the distribution continues shrinking to zero. It is worthy to note, however, that variance $\hat{V}ar(t)$ slows down its growth at rather high values of the intensity of default probability $IPD(t)$ and, therefore, the further development of the variance, though theoretically interesting, has no practical value. The system described with Equations (6.1)-(7.1) is an isolated system, whose variance $VAR(t)$ grows infinitely as a concave-up function. The skewness $\hat{S}(t)$ and $S(t)$ also behave differently: $\hat{S}(t)$ first declines from zero to negative values because of the left distribution deformation, but as the absorption at the boundary proceeds, $\hat{S}(t)$ gradually grows to small positive values. In the end of its evolution, the skewness returns to zero as the distribution finally disappears because of the boundary losses. The skewness $S(t)$ always grows from zero to negative values due to the ever-expanding distribution's left tail.

Returning to jump-diffusion processes discussed in Introduction, one can see that the jump part of JDPs

tries to compensate the difference between the high EMM default probability produced by a heavy negative tail of the value distribution and the low GBM default probability caused by a light normal tail. To get information about statistical parameters of jumps, JDPs use heuristic methods for collecting this information from the market. As a result, jump parameters turn out to be market-averaged ones taken at a fixed moment, but EMM model shows directly that the distributionis dynamic and its development depends mostly on the state of the firm and current market parameters such as the volatility, interest rate, inflation rate, taxation rate, etc.

It is clear that for a concave-down function of mean returns $H(t)$, the *mean year returns* make a monotone non-increasing function of time, making the firm's stock price to decline. In these conditions, no time-sequence of firm's stock prices can make a martingale, or the martingale (risk neutral) measure does not exist. Later we shall see that for the firms with high H_0 values, mean year returns remain approximately constant for their specific times $t \leq T_{NA}$, and the martingale measure exists in time intervals $(0, T_{NA})$. At that, the lesser H_0 , the shorter is the interval where the martingale measure exists, if any.

As one can see from the development of a firm with continuous payments, the payments introduce inhomogeneity into the problem space, and this development runs differently when firm's initial parameters H_0 and VAR_0 are varied (see the remark about Equation (1a) in Introduction). Now let us consider the case when a firm pays its BSEs of size P_k , $k = 1, 2, 3, \dots$ in the end of each k -th time unit (a year)—the Second extended Merton model—EMM2.

In the open logarithmic space ($z = \ln(X/\langle X_0 \rangle)$, $-\infty < z < \infty$), an equation for the firm value development between the payments is

$$dz = Rdt + CdW, R = \alpha - C^2/2, \quad (2.2)$$

an equation for the probability distribution with the normal initial condition $U_0(z; H_0, \sigma_0^2)$ is

$$\begin{aligned} U_t + RU_z - 0.5C^2U_{zz} &= 0, \\ U(z, 0) &= U_0(z; H_0, \sigma_0^2), \end{aligned} \quad (3.2)$$

and the probability distribution before payment ($0 \leq t \leq 1$) remains normal:

$$\begin{aligned} U(z, t) &= (2\pi\sigma^2)^{-1/2} \exp \{ -(z - H)^2 / (2\sigma^2) \} \\ H(t) &= H_0 + Rt, \sigma^2(t) = \sigma_0^2 + C^2t. \end{aligned} \quad (4.2)$$

The effect of one-time BSE payment P_k consists in an instant left shift LP_k of each point of $U(z, t)$, (4.2):

$$LP_1(z) = \ln [(X - P_1)/X], X \geq P_1.$$

The payment P_1 makes a boundary separating firms who survived the payment from those who defaulted at the payment. In z -space the default boundary is $BP_1 = \ln(P_1/\langle X_0 \rangle)$. The BP_1 -boundary distribution function $\tilde{U}(z, 1)$ for the survived firms is

$$\begin{aligned} \tilde{U}(z, 1) &= U(z, 1) - U(2BP_1 - z, 1), \\ \ln [(X - P_1)/X] &= \ln (1 - e^{BP_1 - z}), \end{aligned} \quad (5.2)$$

and the transformed function can be written as

$$z^{(1)} = z + \ln (1 - e^{BP_1 - z}), \quad (6.2)$$

$$U_1(z^{(1)}, 1^+) = (1 - e^{BP_1 - z}) \tilde{U}(z, 1^-), \quad (7.2)$$

$$BP_1 \leq z < \infty, -\infty < z^{(1)} < \infty,$$

$t = I^-$ and I^+ are the moments in the end of the first year *before* and *after* the P_1 payment, the factor $1 - e^{BP_1 - z}$ in Equation (7.2) provides for the integral identity $U(z^{(1)}, 1^+) dz^{(1)} = \tilde{U}(z, 1^-) dz$ because the number of firms in an elementary volume dz around the point z must remain the same after the transformation.

This transformation affects differently high ($X_1 \gg P_1$) and low ($X_1 \cong P_1$) values of the distribution: high values shift insignificantly while low values travel large distances to the left adding negative skewness to the distribution. This skewness increases the default probability for all firms, and brings to imminent default the firms with $X_1 \leq P_1$.

The probability distribution for time t before the second payment, $1^+ \leq t \leq 2^-$, is

$$U(z, t) = (2\pi C^2(t-1))^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left[-\frac{(z-Rt-\xi)^2}{2C^2(t-1)}\right] U_1^+(\xi) d\xi \quad (8.2)$$

$$U_1^+(z) \equiv U_1(z, 1^+). \quad (9.2)$$

At the end of the year, $t_k = 2^+$, $k = 2$, the distribution undergoes another shift

$$LP_2(z) = \ln[(X - P_2)/X], \quad X \geq P_2.$$

For payment P_2 we define boundary $BP_2 = \ln(P_2/X_0)$ and the BP_2 -boundary distribution function $\tilde{U}(z, t)$ (here function $U(z, t)$ is determined by equation (8.2) at $t = 2$)

$$\tilde{U}(z, 2^-) = U(z, 2) - U(2BP_2 - z, 2), \quad (10.2)$$

$$z^{(2)} = z + \ln(1 - e^{BP_2 - z}), \quad (11.2)$$

$$U_2(z^{(2)}, 2^+) = (1 - e^{BP_2 - z})\tilde{U}(z, 2^-); \quad (12.2)$$

$t = 2^-$ and 2^+ are the moments at the end of the second year *before* and *after* the payment. In Equations (11.2) and (12.2), one has $BP_2 \leq z < \infty$, $-\infty < z^{(2)} < \infty$.

The probability distribution for time t , $2^+ \leq t \leq 3^-$, before the third payment, is

$$U(z, t) = (2\pi C^2(t-2))^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left[-\frac{(z-Rt-\xi)^2}{2C^2(t-2)}\right] U_2^+(\xi) d\xi \quad (13.2)$$

$$U_2^+(z) \equiv U_2(z, 2^+). \quad (14.2)$$

Continuing this procedure further, one can find the probability distribution for any time t and a sequence of payments P_1, P_2, P_3, \dots . Each payment increases the distribution negative skewness more and more.

The distribution mean $H_d(t)$, here subscript d stands for “discrete payments”, develops linearly, $H_d(t) = H_0 + Rt$, in the interval $0 \leq t \leq 1^-$. At $t = 1^+$ the distribution instantly gets left (negative) skewness which leads to a leap down in the mean value $H_d(t)$. In the next time interval $1^+ \leq t \leq 2^-$, $H_d(t)$ grows with a decreasing rate that makes $H_d(t)$ a concave-down function in this time interval; in the end of the second year the mean undergoes another leap down, etc. As a result of a sequence of payments, the mean $H_d(t)$ becomes a piecemeal function with leaps down at times of payments t_k , $k = 1, 2, 3, \dots$ and segments of a continuous concave-down function between the leaps. At some time depending on the firm’s initial conditions, the mean $H_d(t)$ achieves its maximum and then goes down. It is important that consecutive mean year returns make a non-increasing sequence tending to zero as $H_d(t)$ achieves maximum. After the maximum, the mean year returns become negative as the mean returns tend to zero. The variance $VAR_d(t)$ grows linearly during the first year $0 \leq t \leq 1^-$: $VAR_d(t) = \sigma_0^2 + C^2t$. At $t = 1^+$, then the variance leaps up as a result of payment P_1 and the instant distribution expansion to the left. Within the next year $1^+ \leq t \leq 2^-$, the variance $VAR_d(t)$ monotonically grows because of the diffusion spread and further distribution distortion, etc. The variance $VAR_d(t)$ occurs to be a piecemeal function with leaps up at times of payments t_k , $k = 1, 2, 3, \dots$; between the leaps it consists of segments of a concave-up function. The distribution skewness $S_d(t)$ remains zero during the first year, $0 \leq t \leq 1^-$, but at the moment of payment $t = 1^+$, the skewness jumps down to a negative value. In the next year $1^+ \leq t \leq 2^-$, the skewness $S_d(t)$ continuously and monotonically decreases until the moment $t = 2^+$ when it undergoes another jump down, etc. The skewness $S_d(t)$ is a piecemeal function with leaps down at times of payments t_k , k

= 1, 2, 3, ... and consisting of segments of a continuous concave-down function. As one can see, lump BSE payments provide for a growing negative tail to the originally normal distribution, and also introduce sharp changes (leaps) into the log-value distribution and its statistical moments. However, the main findings for development of the firm with continuous payments that its mean and negative skewness are concave-down functions of time remain true in the case of discrete payments, too.

The real firm value distribution evolves in a *semi-open space* constrained from below with an absorbing boundary—the *default line* ($DL \leq z < \infty$, $DL = \ln(X_D/X_0)$, X_D is a firm's debt). The probability distribution $\widehat{U}(z, t)$ is now described by the boundary problem with payments P_k , $k = 1, 2, 3, \dots$ at the end of each k th year

$$\widehat{U}_t + R\widehat{U}_z - 0.5C^2\widehat{U}_{zz} = 0, \quad (15.2)$$

$$\widehat{U}(z, 0) = U_0(z; H_0, \sigma_0^2),$$

$$\widehat{U}(DL, t) = 0. \quad (16.2)$$

Knowing a solution in an open space $U(z, t)$ (4.2), (8.2), (13.2), ..., one can write a solution of the boundary problem as

$$\widehat{U}(z, t) = U(z, t) - U(2DL - z, t). \quad (17.2)$$

Having the distribution $\widehat{U}(z, t)$, $DL \leq z < \infty$, one can compute all desired statistical moments such as the mean $\widehat{H}_d(t)$, variance $\widehat{V}ar_d(t)$, and skewness $\widehat{S}_d(t)$ using Equation (16.1) and substituting function $\widehat{U}(z, t)$ for $\widehat{V}(z, t)$. One can compute the intensity of default probability $IPD_d(t)$ with Equation (13.1) after substituting $U(x, t)$ from Equation (13.2) for $V(x, t)$.

Considering the case of a firm that pays its BSEs in a lump sum P_k at the end of the k th year, $k = 1, 2, \dots$, let us compare the time dependences of $\widehat{H}_d(t)$ and $H_d(t)$, $\widehat{V}ar_d(t)$ and $VAR_d(t)$, and $\widehat{S}_d(t)$ and $S_d(t)$ for the boundary problem (15.2), (16.2) and for the problem (3.2) in the open space, correspondently. For discrete payments P_k , the mean values $\widehat{H}_d(t)$ and $H_d(t)$ behave very similarly; they are both piecemeal functions with leaps down caused by yearly lump payments. Between the leaps, $\widehat{H}_d(t)$ and $H_d(t)$ consist of segments of concave-down functions of time: $\widehat{H}_d(t)$ because of losses at the boundary, and $H_d(t)$ due to the ever-growing negative (left) tail. The main difference between them is in the time-scales of their development; because of losses, $\widehat{H}_d(t)$ achieves its maximum and then degrades to zero sooner than $H_d(t)$.

Variances $\widehat{V}ar_d(t)$ and $VAR_d(t)$ having common traits still behave differently. Both functions $\widehat{V}ar_d(t)$ and $VAR_d(t)$ are piecemeal functions with leaps at the times of payments due to instant distribution deformations. However, in the first part of its evolution, $\widehat{V}ar_d(t)$ is an increasing concave-up function of time growing due to the diffusion expansion and the distribution deformation. By and by, its growth slows down to zero as $\widehat{V}ar_d(t)$ achieves its maximum. Finally, variance $\widehat{V}ar_d(t)$ drops to zero while the distribution continues its shrinking to zero in the last part of its evolution. In the time interval where the variance increases, $\widehat{V}ar_d(t)$ leaps up at the moments of payments. The length of those leaps decreases slowly to zero as function $\widehat{V}ar_d(t)$ approaches its maximum. In the time interval where the variance declines, function $\widehat{V}ar_d(t)$ leaps down at the moments of payments. The leap length first increases, then fades to zero. Again, variance $\widehat{V}ar_d(t)$ slows down its rise at high values of the intensity of default probability $IPD(t)$, thus, the further development of the variance has no practical value. The problem (3.2) considers an isolated system, whose variance $VAR_d(t)$ grows infinitely with concave-up segments between the leaps up. The length of those leaps gradually declines to zero. The skewness functions $\widehat{S}_d(t)$ and $S_d(t)$ also behave differently: $\widehat{S}_d(t)$ first declines from zero to negative values because of the distribution deformation. As the absorption at the

boundary proceeds, $\hat{S}_d(t)$ by and by becomes positive. At the end of its evolution, the skewness $\hat{S}_d(t)$ returns to zero as the distribution finally disappears due to losses. The skewness $S_d(t)$ always declines from zero to lesser and lesser negative values due to the ever expanding distribution's left tail.

Within any k th year, the effective *rate of year returns* does not remain constant but is a decreasing monotone function of time both for the cases of continuous and discrete payments. It makes the firm's stock price to decrease over time, and, consequently, the time-sequence of stock prices can never be a martingale. In other words, the martingale measure does not exist in these conditions. However, the rise of mean value, caused by the initial positive drift, can counteract to some extent the effect of diffusion mass transfer across the boundary, slowing down skewness development. For sufficiently high H_0 values, mean year returns can remain approximately constant in a time interval $(0, T_{NA})$, and the martingale measure exists in that interval. Vice versa, the closer H_0 to the zero line, the faster runs the distortion of the initially normal distribution, the shorter is the time interval $(0, T_{NA})$ where the martingale measure exists, if any.

According to the *First Fundamental Theorem of Asset Pricing* (Shiryaev, 1998), a (B, S) -market determined in a filtered probability space $(\Omega, F, (F_n), P)$ consists of a bank account $B = (B_n)$, $B_n > 0$ and a finite number d of assets $S = (S^1, S^2, \dots, S^d)$, $S^i = (S_n^i)$. The market operates at time moments $n = 0, 1, \dots, N$, $F_0 = \{\emptyset, F\}$, $F_N = \{F\}$.

The (B, S) -market is a no-arbitraging market if and only if there is a martingale (risk-neutral) measure \tilde{P} equivalent to P -measure, and d -dimensional calibrated sequence

$$\frac{S}{B} = \left(\frac{S_n}{B_n} \right), S_n = (S_n^1, S_n^2, \dots, S_n^d) \quad (18.2)$$

is a \tilde{P} -martingale, that is, for any $i = 1, 2, \dots, d$ and $n = 0, 1, \dots, N$, one has

$$E_{P\sim} \left| \frac{S_n}{B_n} \right| < \infty, E_{P\sim} \left(\frac{S_n}{B_n} \middle| F_{n-1} \right) = \frac{S_{n-1}^i}{B_{n-1}^i} \quad (19.2)$$

here $E_{P\sim}$ means an operation of taking average of a random variable in the martingale (risk-neutral) \tilde{P} -measure.

One can find the Fundamental Theorem of Asset Pricing in a bit different wording in an excellent textbook (Financial Economics 1998, p. 525). A necessary condition for getting (19.2) is a self-financing portfolio, which means that all portfolio incomes go for future investments only; there is no other outflow from that portfolio (Harrison & Kreps, 1979). Using the martingale terminology, one can say that at the market of firms with payments, the calibrated stock price sequence (S_n/B_n) makes a *local martingale* in the time interval $(0, T_{NA})$. Using this theorem, one can conclude that the market for which there is no risk neutral measure is *not* the no-arbitraging market. This explanation shows that neither the model with discrete-time payments nor the model with continuous-time payments supports the ideas of risk neutral probabilities and no-arbitraging markets in general. The no-arbitrage pricing principle can hold only for individual stocks in their specific time intervals $t \leq T_{NA}$, where the firm's mean year returns remain constant.

General understanding of the process (5.1) with continuous BSE payments and the process (2.2) with discrete BSE payments sheds light on the behavior of the intensity of default probability $IPD(t)$. Because the process drives the Brownian particles to negative infinity, and the faster, the deeper their locations under the line $x = 0$, the intensity of default probability $IPD(t)$ grows with an acceleration over time (a growing concave-up function). For the problem (2.2) with discrete BSE payments, the function $IPD(t)$ acquires additional leaps up at the moments of BSE payments. The default probability PRD as an integral of $IPD(t)$ over some time interval is a continuously increasing monotone function of time.

We solve the problem of EMM1(6.1), (7.1), (10.1) with $\pi(t) \equiv 1$ numerically, estimating the mean returns $H(t)$, variance $VAR(t)$, skewness $S(t)$, and the intensity of default probability $IPD(t)$ for x -distribution. We trace down a dependence of the default probability on factors of the firm and its business environment. Knowing the risk threatening a firm is important not only for banks issuing commercial credits to the firm, or for the companies insuring those credits, but also for the firm's management when planning long-term business operations. At this stage, we suppose that all *perfect market assumptions* hold. We present examples of modeling of distribution $V(x, t)$ and its statistical moments $H(t)$, $VAR(t)$, $S(t)$ in Figures 1-4, and the intensity of default probability $IPD(t)$ for different initial conditions in Figure 5. Model parameters are $R = 0.10$, $\sigma_0^2 = 0.03$, $C^2 = 0.008$, $T = 10$ (years), $DL = 0$.

Table 1

Relation Between the Initial Value H_0 and a Share of BSEs in Mean Year Returns

H_0	$\frac{P}{R\langle X_0 \rangle}$	H_0	$\frac{P}{R\langle X_0 \rangle}$	H_0	$\frac{P}{R\langle X_0 \rangle}$	H_0	$\frac{P}{R\langle X_0 \rangle}$
4.0	0.0183	1.4	0.248	0.9	0.407	0.4	0.670
3.0	0.050	1.3	0.273	0.8	0.449	0.3	0.741
2.5	0.082	1.2	0.301	0.7	0.497	0.2	0.819
2.0	0.136	1.1	0.333	0.6	0.549	0.1	0.905
1.5	0.223	1.0	0.368	0.5	0.607	0.	1.

Figures 1a, 1b show the difference $H(t)-H_0$ as a function of time and H_0 . Table 1 explains the choice of H_0 values illustrating a relation between H_0 and the BSE share in average year returns $P/(R\langle X_0 \rangle)$. We include the first two entries in the table ($H_0 = 4.0$ and 3.0) corresponding to low BSE shares to demonstrate that the difference $H(t)-H_0$ for x -distribution tends asymptotically to the GBM difference $H(t) - H_0 = Rt$, as H_0 tends to infinity.

Figures 1a, 1b supporting the inference of our qualitative analysis demonstrate that all lines $H(t)$ fall apart into two classes: the class of $H(t)$ -lines first rising then falling, and the second class of $H(t)$ -lines falling from the start. The second class is of no practical interest; we do not consider it here. The main parameter controlling this division is H_0 -parameter; other problem parameters (R , σ_0^2 , and C) contribute to a lesser degree. The critical H_0 , separating the classes, is about a unit for the chosen problem parameters. Among the rising and falling lines, there are the lines whose rise takes a long time (decades, lines 1-6). One can consider such lines as semi-steady ones in not too long time intervals. Figure 1a shows that a slope of each $H(t)-H_0$ line declines from R -value specific for GBM as H_0 descends from high to low values. The $H(t)$ -lines with H_0 in the interval (1.10, 1.20) are the lines of stagnation for whom $H(t)$ varies within $\pm 0.1 H_0$ in the period of ten years. Figure 1a proves that the GBM mean (line G) makes no good approximation for the EMM1 lines $H(t)$. Essentially, that all $H(t)-H_0$ curves are concave-down (like lines 7 and 8), and sooner or later, all lines $H(t)$ shall pass their maximum and go down.

The approximately straight rise of $H(t)$ for high H_0 (lines 3-6), providing for almost constant mean year returns, says that the no-arbitrage pricing principle is effective for the assets (stocks or bonds) issued by the firms within those time intervals. However, this principle is not a characteristic of the market, but rather a characteristic of an individual asset and the firm standing behind it. Figure 1b shows clearly the decrease in mean year returns for the firms with $H_0 = 1.20$, 1.15 , and 1.10 . Mark that line 1 ($H_0 = 1.20$) remains straight for eight years, line 2 ($H_0 = 1.15$) remains straight for four years, and line 3 is never straight. Correspondingly,

stock prices of the firm 1 ($H_0 = 1.20$) are time-invariant for eight years, stock prices of the firm 2 ($H_0 = 1.15$) are time-invariant for four years, and stock prices of the firm 3 ($H_0 = 1.10$) always decline. So, the no-arbitrage pricing principle holds for the stock-1 for 8 years ($T_{NA} = 8$), for the stock-2 for 4 years ($T_{NA} = 4$), and it is never holds good for the stock-3 ($T_{NA} = 0$). However, a growing volatility introduces its corrections to lengths of the time intervals where the stock prices remain constant.

Of course, real T_{NA} estimates are determined by the boundary problem $\hat{H}(t)$ solutions:

$$\hat{H}(t) = \int_{DL}^{\infty} x \hat{V}(x, t) dx \quad (20a.2)$$

$$\hat{H}_d(t) = \int_{DL}^{\infty} x \hat{U}(x, t) dx \quad (20b.2)$$

here $\hat{V}(x, t)$ is the solution of the problem (6.1)-(8.1) with continuous payments, and $\hat{U}(x, t)$ is the solution of the problem (15.2)-(16.2) with discrete payments. Real T_{NA} values are shorter than the values estimated above.

The x -variance VAR (Figure 2) demonstrates the pattern also supporting the qualitative analysis. When H_0 descends from high to low values, the variance grows fast, starting from the Ct -line specific for the GBM-distribution. For small t near the start where x -distribution skewness is still low, the variance is close to its GBM-approximation. When x -distribution gains material skewness, its variance significantly exceeds the GBM-variance. Variance values for the cases with low H_0 in the end of the ten-year period show violent fluctuations: for $H_0 = 1.1$, the standard deviation increases from $\sigma_0 = 0.17$ to $\sigma = 0.69$ ($VAR = 0.47$), while for GBM-distribution the standard deviation is half as much ($VAR = 0.11$, $\sigma = 0.33$). The excess of x -variance over GBM-variance is due to the distribution deformation. To the contrary of the GBM-variance having constant volatility C , x -variance has a time-varying effective volatility $C_{eff}(0) = C$, $C_{eff}(t) = [dVAR/dt]^{1/2}$, which is an increasing monotone function of time. The growing volatility shortens the time intervals where the stock prices can be constant (Figures 1a and 1b).

We see a similar pattern in the development of x -distribution skewness (Figures 3a and 3b). From small negative values specific to high H_0 (4.0 or higher), it declines fast, achieving large negative values of about -0.5 ($H_0 = 1.1$). For the standard skewness S_{st} :

$$S_{st} = S/VAR^{3/2}$$

and H ($t=10$, $H_0 = 1.1$), one has $S_{st} = -1.33$. The S&P500 Index's average value for 1970-2000 is $S_{st} = -1.73$ (Kou, 2007). Figure 4 shows typical examples of x -distribution, where one can observe the development of longer negative tails.

Figures 5a and 5b show how the intensity of default probability $IPD(t, H_0)$ depends on time and initial H_0 . Mark that development of $IPD(t, H_0)$ is very inertial: for a significant part of 10 years, IPD remains low, rising to noticeable values in the second half of the graph. Figures 5a and 5b demonstrate a fast IPD rise when H_0 decreases. This behavior of the $IPD(t)$ function can suggest a false feeling of safety to the management of the firm using a one-year horizon model like the Moody's KMV (Bohn, 2006) for estimating a firm's state. Such models warn on a coming crisis too late, when a significant part of the time, the management needs to improve the situation, is already lost.

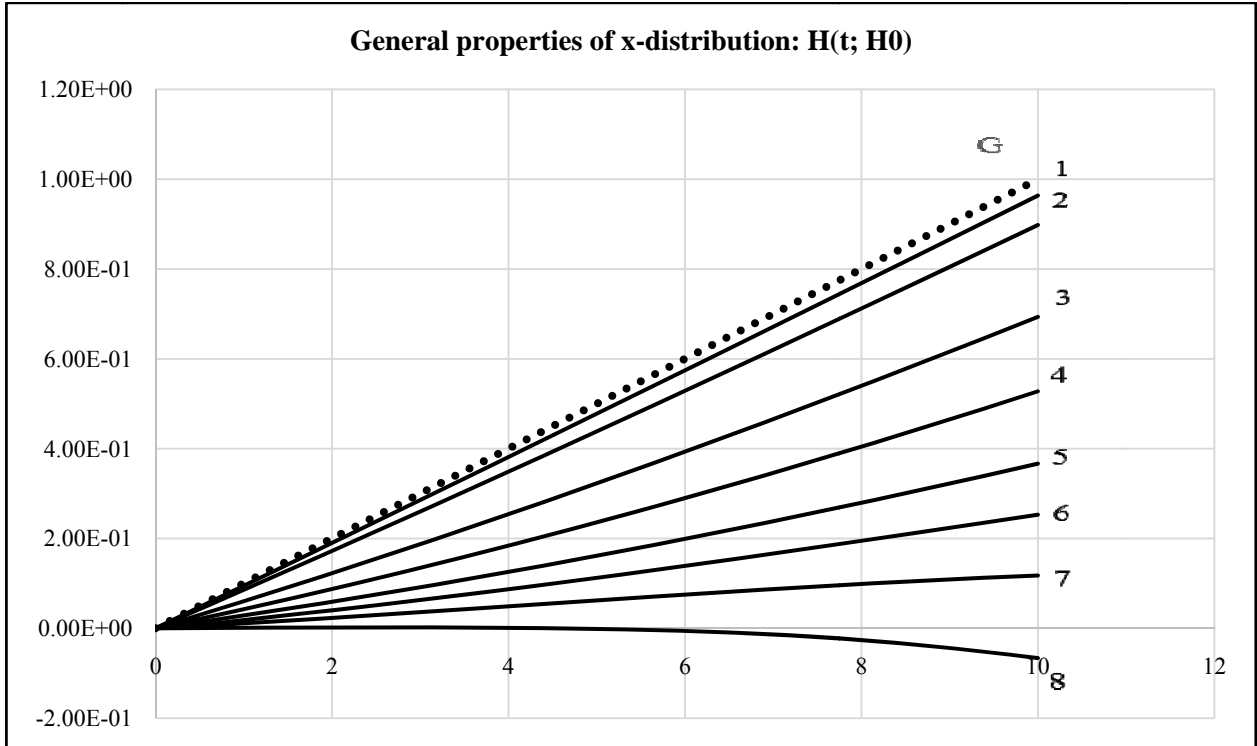


Figure 1a. Log-value mean $H(t; H_0) - H_0$ as a function of time (years) and initial H_0 values: $H_0 = 4.0(1), 3.0(2), 2.0(3), 1.6(4), 1.4(5), 1.3(6), 1.2(7), 1.1(8)$. The dot line (G) shows a drift of the asymptotic GBM solution $\Delta H = Rt$.

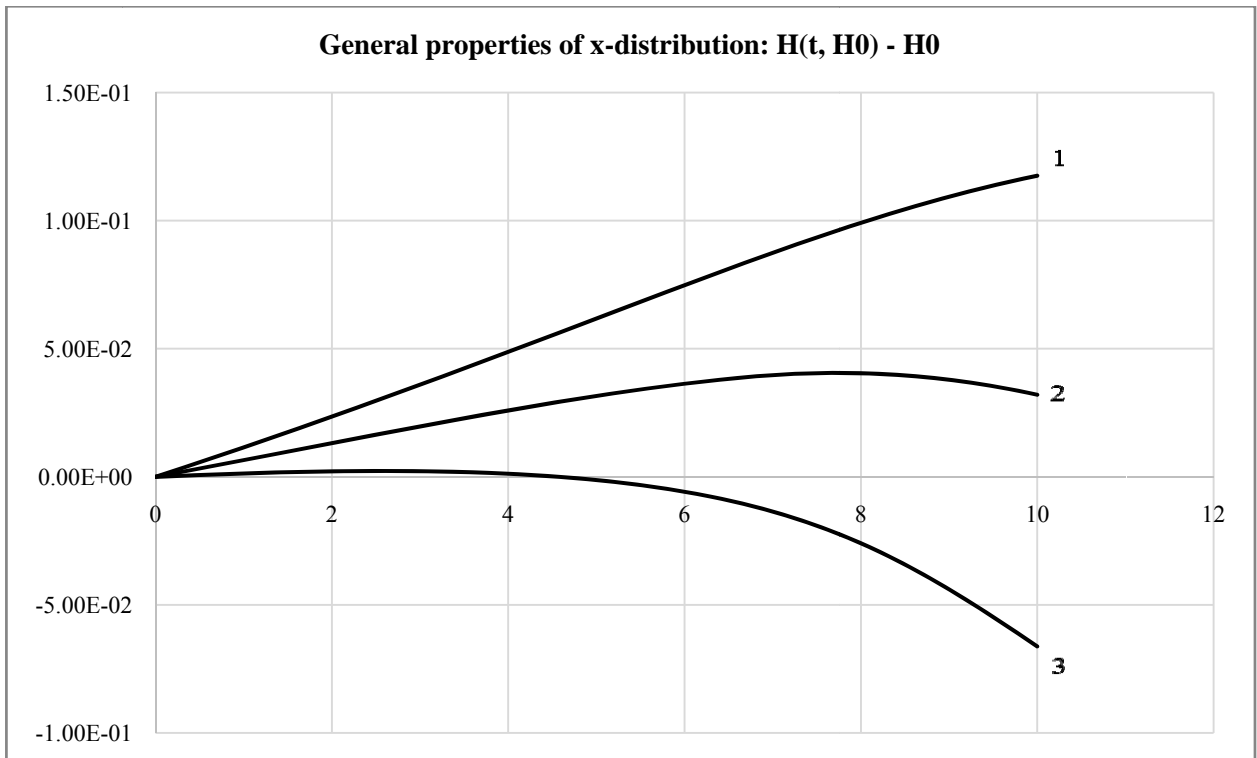


Figure 1b. Log-value mean $H(t, H_0) - H_0$ as a function of time (years) and initial H_0 values. $H_0 = 1.20(1), 1.15(2), 1.10(3)$.

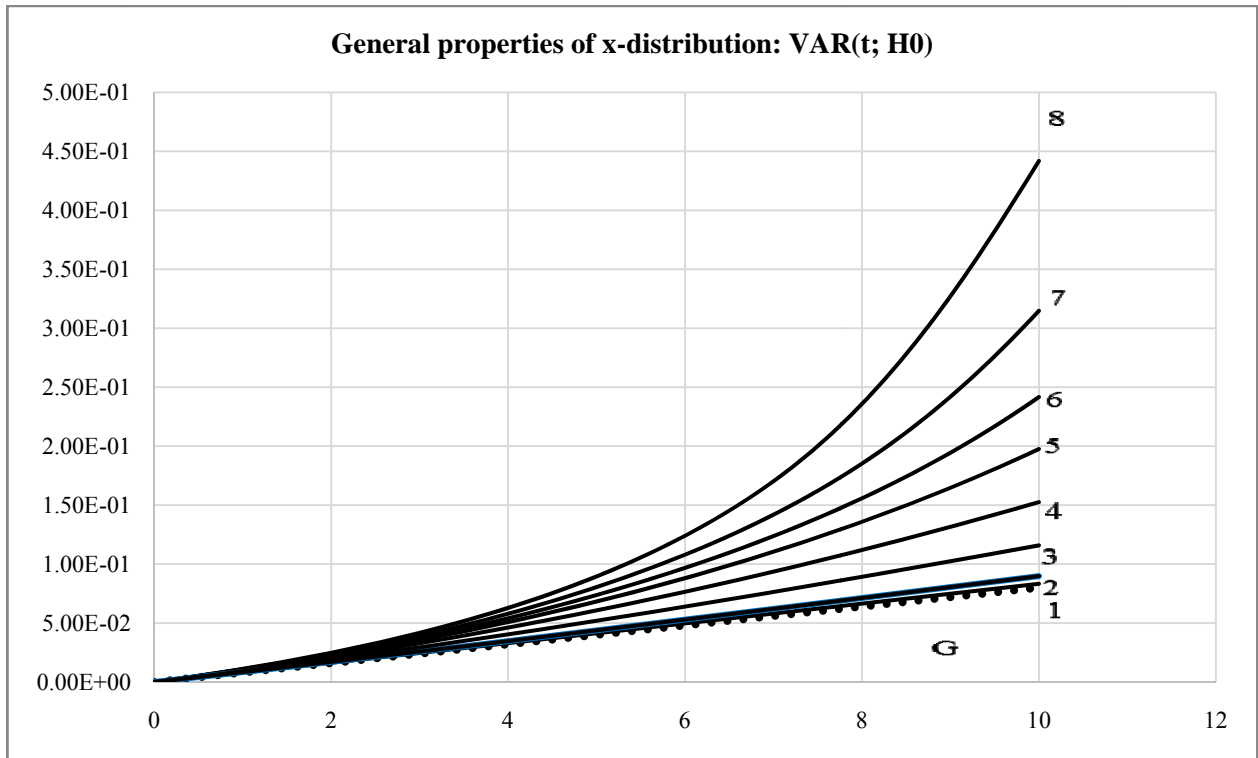


Figure 2. The difference $VAR(t; H_0) - VAR_0$ as a function of time (years) and values $H_0 = 4.0$ (1, dash), 3.0 (2), 2.0 (3), 1.6 (4), 1.4 (5), 1.3 (6), 1.2 (7), 1.1 (8). The dot line (G) shows the spread of the asymptotic GBM solution $\Delta VAR = Ct$.

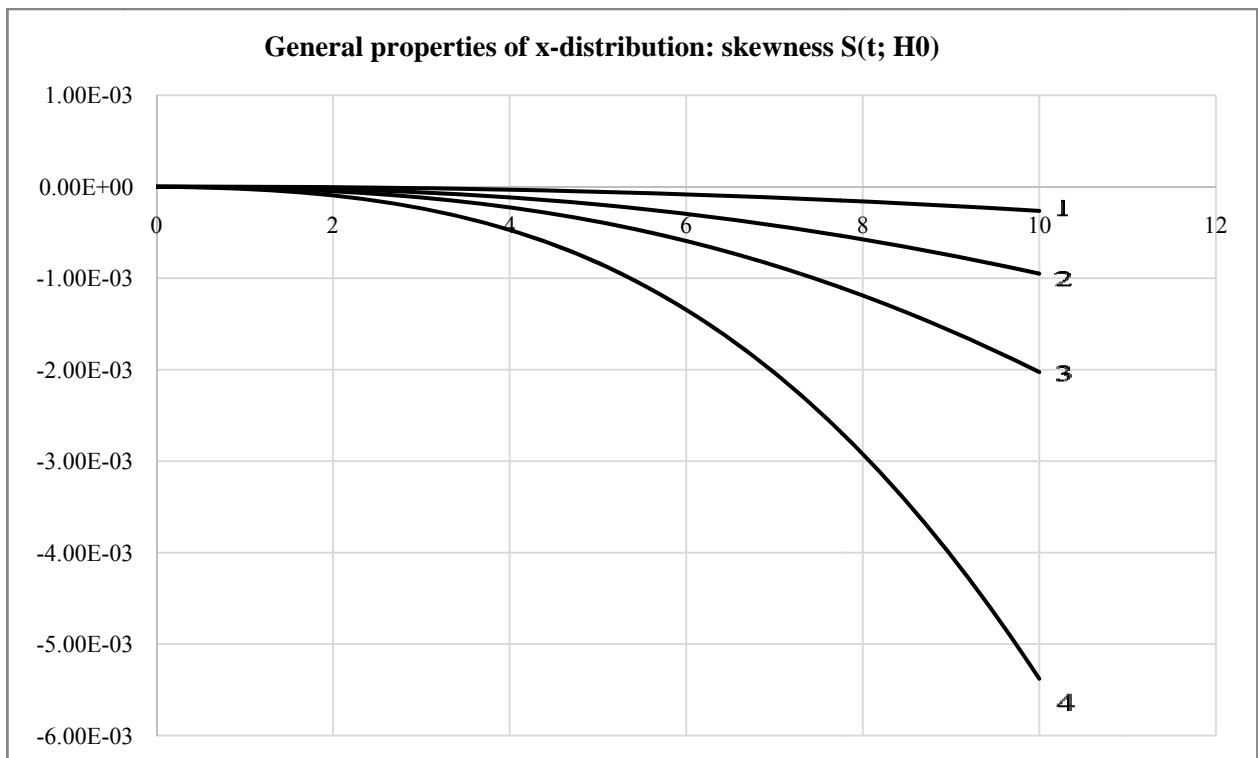


Figure 3a. Development of the x -distribution skewness S as a function of time (years) and values $H_0 = 4.0$ (1), 3.0 (2), 2.0 (3), 1.6 (4).

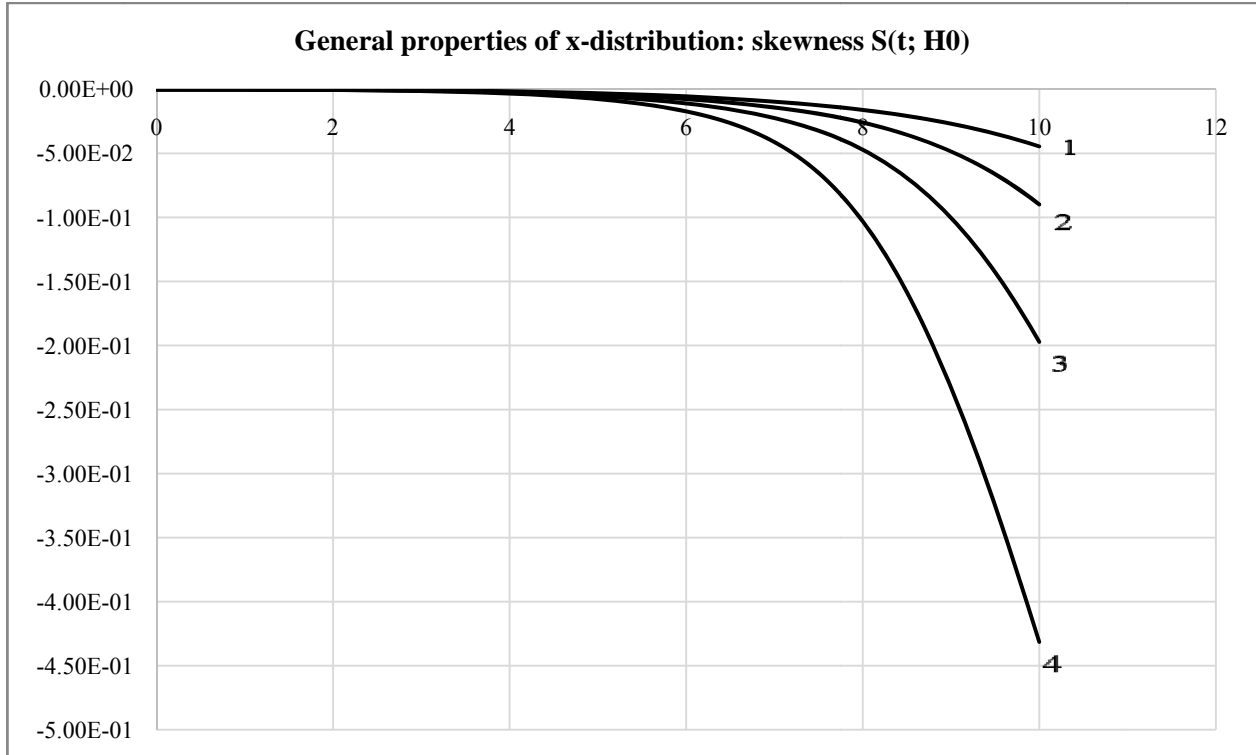


Figure 3b. Development of the x -distribution skewness S as a function of time (years) and $H_0 = 1.4$ (1), 1.3(2), 1.2(3), 1.1 (4).

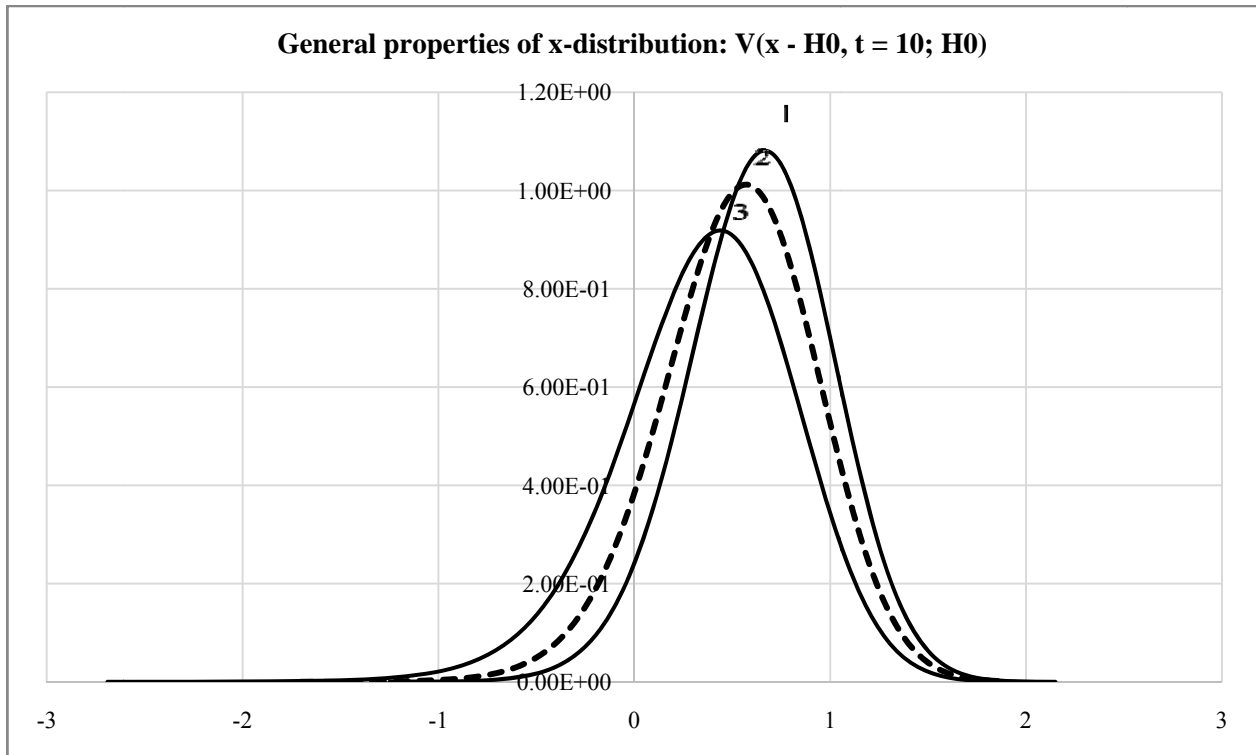


Figure 4. The x -distribution $V(x - H_0, t = 10)$ for $H_0 = 1.8$ (1), 1.6(2), 1.4(3). Observe development of a longer negative tail and increasing distribution skewness.

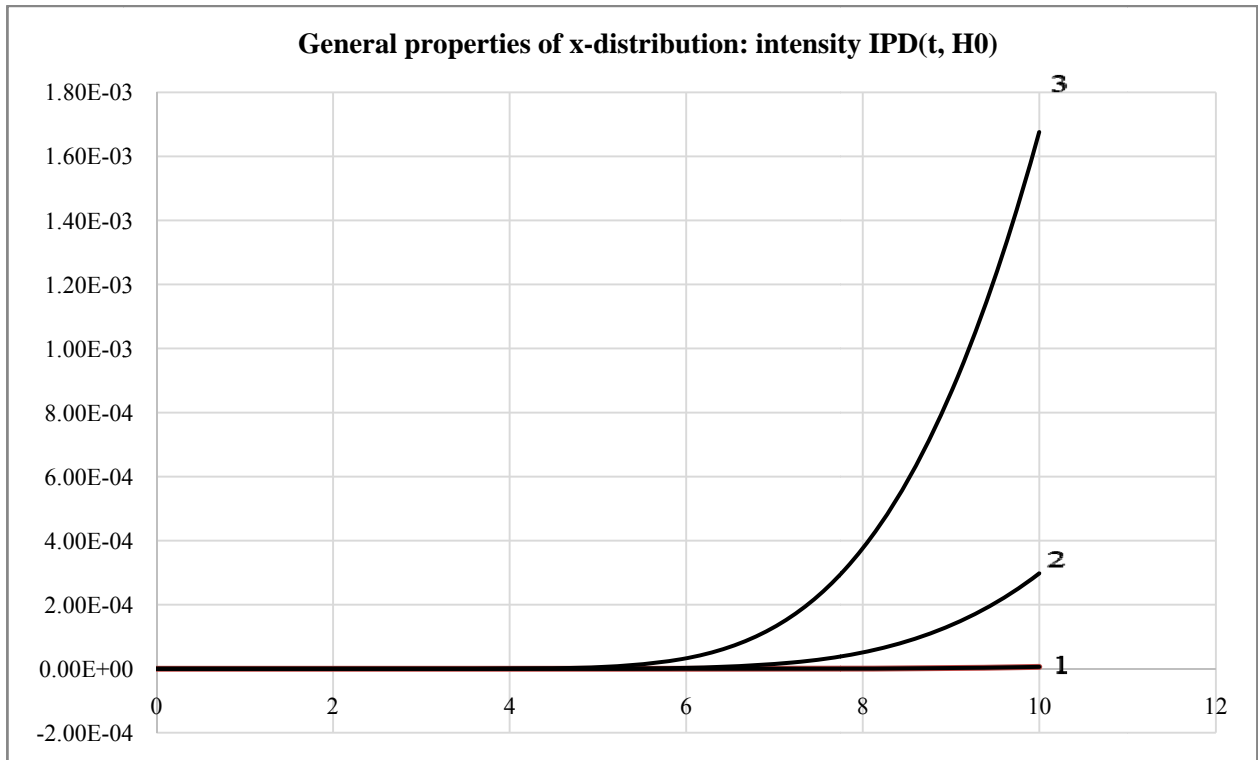


Figure 5a. The intensity of default probability $IPD(t, H_0)$ as a function of time (years) and the initial values $H_0 = 1.80(1), 1.6(2), 1.50(3)$.

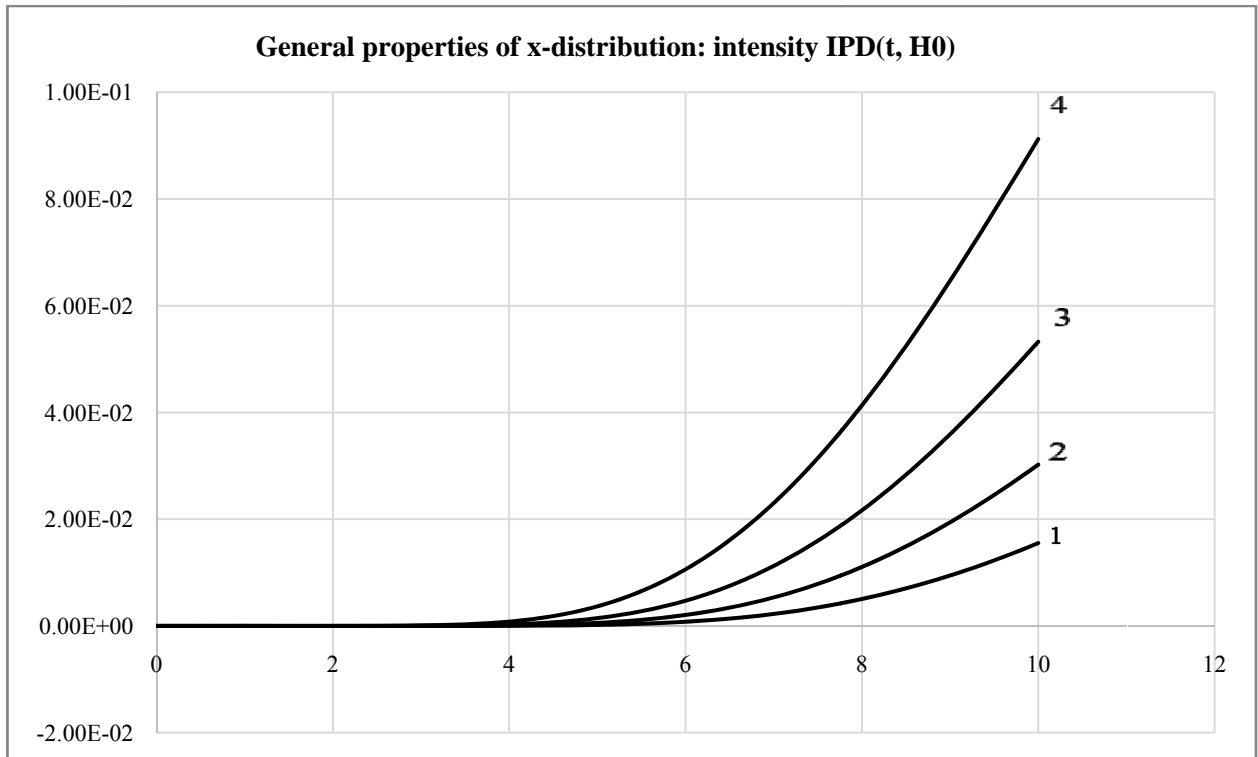


Figure 5b. The intensity of default probability $IPD(t, H_0)$ as a function of time (years) and initial values $H_0 = 1.35(1), 1.30(2), 1.25(3), 1.20(4)$.

The no-arbitraging principle holds for the whole market only if the market consists of the GBM-firms with no payments. For the market consisting of EMM-firms, the no-arbitraging principle holds for each i th stock for the time $t \leq T_{NA}^i$ from the start of the firm's business where the mean year returns remain constant. Therefore, the no-arbitrage pricing principle never holds for the whole market for more or less long times because all firms start their businesses independently at their times. However, for short-term operations with stocks whose time intervals are much lesser than a year, are always non-arbitraging; the expected return of the short-term deal is zero. Considering long-term investments ($T_{inv} \approx T_{NA}^i$) when mean year returns begin to decline, an investor must be very cautious because the martingale characteristic of stock prices and no-arbitrage pricing principle become now ineffective. Increases in debt leverage, tax rate, interest rate, or inflation rate can significantly decrease the firm mean year returns inflicting losses to the long-term investors.

The principal difference between the lognormal GBM-distribution and the skewed EMM-distribution is the following. The GBM-firm now generally used in the financial analysis (see the literature review) remains "ever young" keeping invariable mean year returns R and volatility C ; its default probability is symmetric and grows slowly, and the firm dies (defaults) accidentally amid full prosperity. According to properties of the diffusion motion, the probability that a diffusion process starting at time $t = 0$ from point M on a plane (x, t) shall cross an arbitrary straight line $x = a$ in that plane at a finite time $T(M, a)$ is unit almost for sure: $P(T(M, a) < \infty) = 1$ a.s. (Shiryayev, 1998, pp. 302-303). If line $x = a$ is a default line, then the firm's longevity is finite almost for sure. From an optimistic point of view, that means that there is a set of firms of a null measure whose longevity is infinite: $T(M, a) = \infty$; in other words, there are firms which can exist forever! To the contrary of the ever-young GBM-firm, the EMM-firm by and by "grows old": its *effective mean year returns* $R_{eff}(t)$, $0 < R_{eff}(0) < R$, is a non-increasing monotone function of time, its effective volatility $C_{eff}(t) = [dVAR/dt]^{1/2}$, $C_{eff}(0) = C$, is an increasing function of time, and its negative skewness is an ever decreasing function of time. The left (negative) tail of the distribution grows fast significantly increasing the default probability compared to the GBM default probability. No later than at time T_{Max} (see Equation (23.1)):

$$T_{Max} = \min \left(\begin{array}{l} t_R, t_C; t_R: t < t_R, R_{eff}(t) > 0; R_{eff}(t_R) = 0; \\ t_C: t < t_C, C_{eff}(t) < C_{cr}; C_{eff}(t_C) = C_{cr} \end{array} \right), \quad (21.2)$$

investors will lose their interest in such a firm, get themselves free of the firm's stocks dropping down the stock price and the firm value, and then the firm shall soon default (here C_{cr} is a critical volatility value at which investors recognize the risk of holding the firm's stocks as unacceptable). A diffusion walk of the firm value is now faster ($C_{eff}(t) \geq C$) and asymmetric: a negative move in the firm value is more probable than a positive move. This asymmetry increases the default probability of the EMM-firm compared to the default probability of the GBM-firm and makes the EMM-firm default accidentally any time before T_{Max} . The EMM-firm's longevity is finite, and lesser than the longevity of a comparable GBM-firm.

The firm's business longevity depends on the initial conditions and competition of growth rates of the variance and skewness on the one side, and the positive drift rate on the other. So, just having a positive drift rate is not enough for safe corporate development, the drift rate must be sufficiently high. Any stagnation, leaving aside the decrease in the firm value, affects negatively the firm's survival. For example, for a typical small firm, its longevity is rather short because of a low H_0 (following from an asset shortage), a relatively large initial variance VAR_0 and high sensitivity to market fluctuations C (resulting from a shortage of business skills), and a rather low rate of expected returns R . The situation is further aggravated by debts if any. The firm's business stability depends on the quality of its management and team executing the firm's plans, on a

corporate industry and competition in it, and on a state and trends in the national/global economy. The measures strengthening firm's business health are well known: optimizing payments, raising the rate of expected returns, increasing efficiency of financial management by using the most precise mathematical models, etc. All this is hardly achievable without a regular critical reconsideration of business plans and reorganization of the firm's management, especially the top management as a crucial element of the corporate success. The effects of the firm's management on the firm's longevity are considered in the qualitative corporate life-cycle theory promoted by Adizes (2012).

So far we consider a firm with one business project only. A firm can strengthen its stability by the project diversification developing simultaneously several projects at different stages of their lifecycles starting new projects and closing "aging" projects at optimal times. The theory of large firms running several business projects at a time is still to be developed.

Conclusion

The paper suggests two extensions of the Merton model of 1974 (EMM1 and EMM2) considering BSE payments, and new statistical distributions following from that models. In the open space in variables (x, t) , $x = \ln(RX/P_0)$, X —a firm value, t —time, R —an expected rate of returns, $P = P_0\pi(t)$ is continuous function of payments, the log-value distribution evolves from a normal distribution to a negatively skewed one, with skewness growing over time and mean making a concave-down function of time. In the open space in variables (z, t) , $z = \ln(X/\langle X_0 \rangle)$, X —a firm value, t —time, $\langle X_0 \rangle$ —a mean value of the initial X_0 distribution, $P_k > 0$, $k = 1, 2, \dots$ —discrete lump payments at a k th year, the log-value distribution through a series of leaps at the moments of BSE payments transforms from a normal distribution to a negatively skewed one. Its skewness and mean are piecemeal functions with leaps at the moments of payments, and segments of concave-down functions between the leaps. The principal difference between the GBM-distribution and both EMM-distributions is the following. The GBM-firm remains "ever young" keeping time-invariant mean year returns R and volatility C ; the random walk of the firm value is symmetric, the default probability remains low. The both EMM-firms, to the contrary, gradually "grow old": their effective mean year returns $R_{eff}(t)$, $0 < R_{eff}(0) < R$, are non-increasing functions of time, and their effective volatilities $C_{eff}(t) = [dVAR/dt]^{1/2}$, $C_{eff}(0) = C$ grow over time. The random walk of the firm value for the both models is asymmetric and more intensive than in the GBM case: a negative move in the firm value is more likely than a positive move; this imbalance increases the default probability. The EMM-firm has a lesser longevity than the GBM-firm. EMM helps to analyze the firm's longevity as a function of time and parameters of the firm's business conditions, and choose the best feasible means to strengthen the firm's economic position. Based on the new firm value distribution, we present the continuous-time models computing the firm present value PV and the project net present value NPV with an analysis of factors affecting the time horizons of those models. It is shown that the firm PV and project NPV equations now in use are internally inconsistent and misleading.

The risk-neutral probabilities and no-arbitrage pricing principle follow from the ability of the firm to keep its mean year returns time-invariant. It is always correct at the GBM-market with firms making no payments or with firms whose payments are proportional to their values ($P = \delta X$) because mean year returns of the GBM-firm always equal R ($\delta = 0$) or $R - \delta$ ($\delta \neq 0$). When a model takes account of the BSE payments in a general form (as both EMMs do), then for a sufficiently long time, the firm mean year returns begin to decline. In such conditions the stock price of the firm decreases, too. There are no martingale measures and risk-neutral

probabilities, and the no-arbitraging principle for the market is ineffective for both modes of payment, continuous or discrete. At the market consisting of firms paying their BSEs, a firm can keep its mean year returns (approximately) constant only for some time, $0 \leq t \leq T_{NA}$, depending on its business conditions. In other words, the risk-neutral probabilities and no-arbitraging principle hold for individual firms and only in their specific time intervals $(0, T_{NA})$, rather than for the entire market and for all times as most economists believe now. For short-term deals with $T_d \ll T_{NA}$, when a trader buys an asset and soon resells it trying to profit on the asset price difference, the no-arbitraging principle always holds, and the market is fair to such traders. For long-term investors with investment times $T_{inv} \approx T_{NA}$, the picture is quite different. A class of long-term investors includes such weighty investors as pension funds, mutual funds, insurance companies, banks, and big firms. For such investors the effects of payments are essential, and the investors must timely re-estimate their portfolios because the mean year returns of any firm in the portfolio decrease over time for $t \geq T_{NA}$ (the stocks become “stale”). The martingale characteristic of stock prices and effectiveness of the no-arbitraging principle depend on the parameters of the firm issuing the stock. Increase in debt, interest rate, tax rate, inflation rate, etc. happening within an investment period decrease the firm’s mean year returns inflicting losses to investors.

An alerting implication from our study is that the results derived with the technique of risk-neutral probabilities when analyzing effects of debt and default are misleading for the theory of financial economics and dangerous for practice. The EMM1 and EMM2 models can be helpful to the firm’s management for a better understanding of the firm’s current state and the prospects of its development, especially when planning long-term business operations. It can also be useful for long-term investors keeping firms’ stocks in their portfolios for a long time, and also for banks and insurance companies estimating credit risks for a particular commercial borrower over the debt maturity.

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