

Comparative Analysis of the Deflections of Two Beams Using the Finite Difference Method

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Abstract: In this paper, a comparative analysis of two beams' deflections, one supported-embedded beam and a bi-supported beam, is presented. For such comparison, first the respective second-order linear Ordinary Differential Equations (ODEs) were obtained. Along with the boundary conditions, there are two Boundary Value Problems (BVPs), making it possible to perform their numerical and analytical solutions. For numerical solutions, a Matlab algorithm was implemented based on the Finite Difference Method (FDM). The analytical solutions were also obtained for comparison with the numerical ones and with the validation method. In the end we analyzed the shapes of the elastic lines of the two beams caused by the loads coming from the weight of each one.

Key words: FDM, BVP, ODE, beams, deflection.

1. Introduction

Beams are structural elements used in civil construction. When projected, the maximum vertical displacement they are subject to is taken into account. The study of this displacement is called deflection. And the deformed shape of the beam axis is called the elastic line. The second-order linear Ordinary Differential Equation (ODE) conducts the behavior of the elastic line, as Refs [1, 2] are given by Eq. (1).

$$\frac{d^2u}{dx^2} = \frac{M(x)}{EI} \quad (1)$$

in which u denotes the function that controls the deflection of the beam in a distance x , M the bending moment, I the moment of the cross-section inertia and E is Young's modulus.

1.1 Supported-Embedded Beam

Fig. 1, represents a supported-embedded beam, in which, q is the uniformly distributed load (UDL), l is

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the length of the span and R_a is the support reaction in a .

From a cut in the section represented in Fig. 1, the bending moment equation is found through the static equilibrium. Thus, when replacing in ODE (Eq. (1)), Eq. (2) is found,

$$\frac{d^2u(x)}{dx^2} = \frac{R_a x - \frac{qx^2}{2}}{EI} \quad (2)$$

representing the differential equation of the elastic line for the supported-embedded beam subject to the boundary conditions given in Eq. (3),

$$u(0) = 0 \text{ and } \frac{du}{dx}(l) = 0 \quad (3)$$

such boundary conditions indicate that the displacement is null at the supported left end, while at the embedded right end there is no rotation, indicated by the first derivative. Eqs. (2) and (3) guarantee the solution uniqueness of the boundary value problem (BVP) of the Dirichlet-Neumann type according to Ref. [3].

When integrating Eq. (2) and applying the boundary conditions (Eq. (3)), it is found:

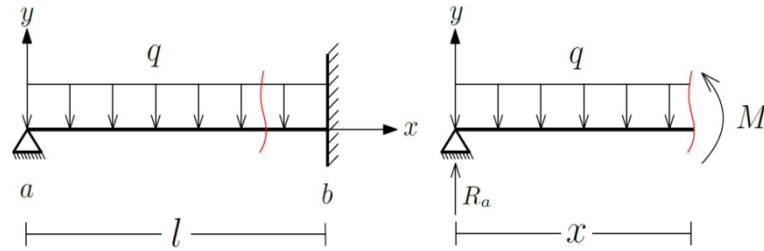


Fig. 1 Supported-embedded beam with a cut at a distance x , where $0 \leq x \leq l$.

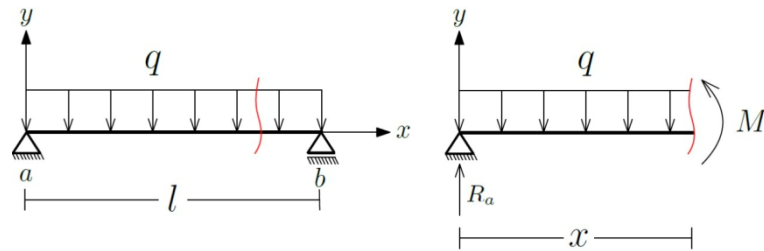


Fig. 2 Bi-supported beam with a cut at a distance x , where $0 \leq x \leq l$.

$$u(x) = \frac{q}{EI} \left(\frac{lx^3}{16} - \frac{x^4}{24} - \frac{xl^3}{48} \right) \quad (4)$$

which is called the analytical solution of the referred beam.

1.2 Bi-supported Beam

In Fig. 2 a bi-supported beam is represented, where l is the length of the span, q is the intensity of the UDL and R_a is the support reaction.

As performed in Section 1.1, the bending moment equation for the beam in question must be obtained to replace it in the ODE (Eq. (1)), resulting in Eq. (5).

$$\frac{d^2u(x)}{dx^2} = \frac{qx}{2EI} (l-x) \quad (5)$$

called the differential equation of the elastic line for the bi-supported beam, subject to the boundary conditions given in Eq. (6),

$$u(0) = 0 \text{ and } u(l) = 0 \quad (6)$$

In this situation, the boundary conditions indicate that the displacements at both supported ends are null. Eqs. (5) and (6) guarantee the solution uniqueness of the BVP of the Dirichlet-Dirichlet type, according to Ref. [3].

Integrating Eq. (5) and using the boundary conditions in Eq. (6), we find,

$$u(x) = \frac{q}{EI} \left(\frac{lx^3}{12} - \frac{x^4}{24} - \frac{xl^3}{24} \right) \quad (7)$$

representing the analytical solution for the indicated beam.

2. Numerical Solution

The numerical method to be used is Finite Difference Method (FDM), which is a method consisted in transforming a differential equation into an equation of differences generating a linear equations system. When solved, it provides the solutions for the described BVPs, respectively in a Dirichlet-Neumann and Dirichlet-Dirichlet type.

Therefore, an algorithm developed in Matlab software based on FDM was created, making the comparison of numerical solutions with analytical solutions, returning a percentage error between them.

3. Results

For the two studied beams we used $q = 2.5 \text{ N/mm}$, $l = 5,000 \text{ mm}$, $I = 2,083 \times 10^9 \text{ mm}^4$ and $E = 26,071.6 \text{ MPa}$. In addition to these, it was considered for numerical solutions, an $N = 5,001$ for the bi-supported beam and an $N = 5,000$ for the supported-embedded beam, obtaining the deflections and the percentage error between the solutions.

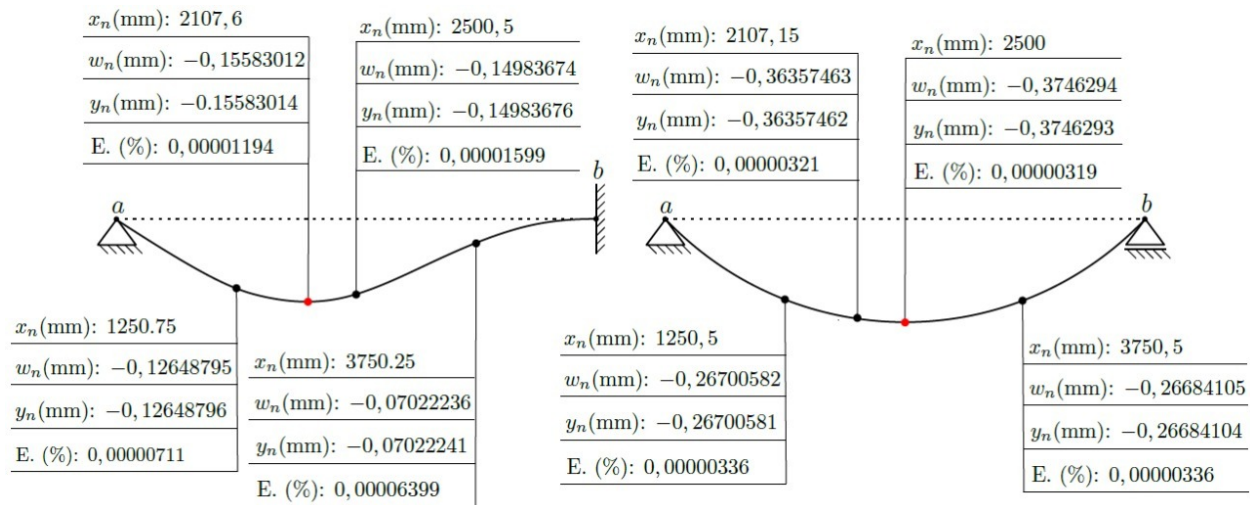


Fig. 3 Deflection at some points along the span of the beams.

Fig. 3 shows some points along the beams' span and their respective deflections. Note that the supported-embedded arrow occurs at $x = 2,107.68$ mm. The red dots represent the arrows of the beams, in other words, are the maximum deflection of each beam and where each one occurs.

4. Conclusions

Comparing the values obtained at the same points for the models of the built beams, it can be observed that the arrow in the supported-embedded beam was smaller than that in the bi-supported one, in addition to its location being closer to the support, showing that the deflection in this case does not happen in a symmetrical way as it happens in the bi-supported

beam. As the crimp does not allow rotation, the beam becomes harder leading to the configuration of the asymmetric elastic line. Thus, the embedding of beams is an appropriate solution for projects with large deflections. Finally, it is noted that the obtained results were satisfactory due to the low percentage error. It was possible to verify the efficiency of the method and the good performance of the algorithm.

References

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