Journal of Mathematics and System Science 9 (2019) 130-135
doi: 10.17265/2159-5291/2019.05.003

On q^2-Trigonometric Functions and Their q^2-Fourier Transform

Sama Arjika[1,2]

1. Department of Mathematics and Informatics, University of Agadez, Post Box 199, Agadez, Niger

2. International Chair of Mathematical Physics and Applications, (ICMPA-UNESCO Chair), University of Abomey-Calavi, 072 B. P.: 50 Cotonou, Benin

Abstract: In this paper, we first construct generalized q^2-cosine, q^2-sine and q^2-exponential functions. We then use q^2-exponential function in order to define and investigate a q^2-Fourier transform. We establish q-analogues of inversion and Plancherel theorems.

Key words: q-bessel function, q-trigonometric function, q^2-fourier transform, inversion theorem, plancherel theorem.

1. Introduction

During the last years, an intensive work was founded about the so-called q-basic theory. Taking account of the well-known Ramanujan works shown at the beginning of this century by Jackson [1, 2], many authors such as Askey, Gasper, Rogers, Andrews, Koornwinder, Ismail, Srivastava, and others (see references) have recently developed this topic.

The present article is devoted to extend the study of the q^2-analogue of the Fourier transforms. The method used here differs from those given by Richard [3]. We take as definition a general form of q^2-cosine [4]:

$$\cos(x;q^2) = \sum_{k=0}^{\infty} \frac{(-1)^k\, q^{k(k+1)}\, x^{2k}}{(q;q)_{2k}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\, x^{1/2}\, J_{-1/2}(x;q^2) \tag{1}$$

and q^2-sine [4]:

$$\sin(x;q^2) = \sum_{k=0}^{\infty} \frac{(-1)^k\, q^{k(k+1)}\, x^{2k+1}}{(q;q)_{2k+1}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\, x^{1/2}\, J_{1/2}(x;q^2), \tag{2}$$

where $J_\alpha(x;\ q^2)$ is the q^2-analogue of the Bessel function [5, 6]

$$J_\nu(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} x^\nu \times {}_1\phi_1\left(\begin{matrix} 0 \\ q^{\nu+1} \end{matrix} \,\middle|\, q;\ q^2x^2 \right). \tag{3}$$

The q^2-analogue Bessel functions and closely related variants have received much attention because of their importance in the study of q-analogues of representations of the Group of Plane Motions and of the Quantum Group of Plane Motions, q-differential equations, and other topics. For more details, see Refs. [7-11].

Our aim in this paper is to give an extension of q^2-analogue trigonometric functions $\cos(x;\ q^2)$; $\sin(x;\ q^2)$ and q^2-analogue exponential function $e(x;\ q^2)$ [4]. We then study generalized q^2-Fourier transform and give the q-analogues of inversion and Plancherel theorems.

The paper is organized as follows: in Section 2, we give notations and preliminaries to be used in the sequel. In Section 3, we define generalized q^2-cosine, q^2-sine and q^2-exponential functions and study some of their properties. We give q-analogues of inversion and Plancherel theorems. We end with concluding remarks in Section 4.

Corresponding author: Sama Arjika, Dr., research field: mathematics.

2. Notations and Preliminaries

Throughout this paper, we assume that $0 < q < 1$; $\alpha > -1$ and we write $\mathbb{R}_{q,+} = \{q^n,\ n \in \mathbb{Z}\}$. We follow the notations and terminology in Refs. [12-14]. The basic hypergeometric series $r\emptyset s$

$$
{}_r\phi_s\left(\begin{array}{c} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{array} \middle| q; x\right) = \sum_{k=0}^{\infty}\left[(-1)^k q^{k(k-1)/2}\right]^{1+s-r} \times \frac{(a_1, a_2, \cdots, a_r; q)_k}{(b_1, b_2, \cdots, b_s; q)_k}\frac{x^k}{(q;q)_k}, \tag{4}
$$

converges absolutely for all x if $r \le s$ and for $|x| < 1$ if $r = s + 1$ and for terminating. The compact factorials of $r\emptyset s$ are defined respectively by:

$$
\begin{aligned}
(a;q)_0 &= 1,\ (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \\
(a;q)_\infty &= \prod_{k=0}^{\infty}(1-aq^k)
\end{aligned} \tag{5}
$$

and

$$(a_1, a_2; \ldots, a_m; q)_n = (a_1; q)_n(a_2; q)_n \ldots (a_m; q)_n \tag{6}$$

where $m \in \mathbb{N} = \{1, 2, \ldots\}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For a complex number x and $n \in \mathbb{N}$, the q-numbers and the q-factorials are defined as follows:

$[x]_q = (1-q^x)/(1-q)$, $[n]_q! = \prod_{k=1}^{n}[k]_q$, $[0]_q! = 1$.

For $\alpha > -1$, we define the generalized q-integers by [15]:

$$
\begin{aligned}
[2n]_{q,\alpha} &= [2n + 2\alpha + 1]_q, \\
[2n+1]_{q,\alpha} &= [2n + 2\alpha + 2]_q
\end{aligned} \tag{7}
$$

and the generalized q-shifted factorials by:

$$(q;\ q)_{n,\alpha} := (1-q)^n [n]_{q,\alpha}!. \tag{8}$$

Remark that, we can rewrite Eq. (8) as

$$(q;q)_{2n,\alpha} = (q^2;q^2)_n (q^{2\alpha+2};q^2)_n$$

and

$$(q;q)_{2n+1,\alpha} = (q^2;q^2)_n (q^{2\alpha+2};q^2)_{n+1}.$$

By means of Eq. (7), we may express the generalized q-factorials as

$$[2n]_{q,\alpha}! = \frac{\Gamma_{q^2}(\alpha+n+1)\Gamma_{q^2}(n+1)}{(1+q)^{-2n}\Gamma_{q^2}(\alpha+1)}$$

and

$$[2n+1]_{q,\alpha}! = \frac{\Gamma_{q^2}(\alpha+n+2)\Gamma_{q^2}(n+1)}{(1+q)^{-2n-1}\Gamma_{q^2}(\alpha+1)},$$

where Γ_q is the q-Gamma function given by [12]:

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z},$$

and tends to $\Gamma(z)$ when q tends to 1^-. In particular, we have the limits:

$$
\begin{aligned}
\lim_{q\to 1^-}[2n]_{q,\alpha}! &= \frac{2^{2n} n!\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)} \\
&= 2^{2n} n!(\alpha+1)_n,
\end{aligned}
$$

where $(a)_k = a(a+1) \ldots (a+k-1)$ is the Pochhammer-symbol [12, 16].

Remark that, for $\alpha = -\frac{1}{2}$, we get:

$$(q;q)_{n,-\frac{1}{2}} = (q;q)_n\ [n]_{q,-\frac{1}{2}}! = [n]_q!.$$

The q-Jackson integrals from 0 to $+\infty$ and from $-\infty$ to $+\infty$ are defined by [1]:

$$\int_0^\infty f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n) \tag{9}$$

and

$$\int_{-\infty}^{\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} q^n[f(q^n) + f(-q^n)], \tag{10}$$

provided the sums converge absolutely.

3. Results and Discussion

In this section, we define and study generalized q^2-trigonometric functions. We then introduce a q^2-Fourier transform that formally tends to its classical analogue as $\alpha = -1/2$, $q \to 1^-$ and study some of its properties.

3.1 Generalized q^2-Analogue Trigonometric Functions

We recall that the q^2-analogue exponential function $e(x; q^2)$ is defined in Ref. [3] by

$$e(x;\ q^2) = \cos(-ix;\ q^2) + i\ \sin(-ix;\ q^2). \tag{11}$$

By means of generalized q-shifted factorials $(q; q)_{2n,\alpha}$ and $(q; q)_{2n+1,\alpha}$, we define generalized q^2-cosine and q^2-sine as follows.

Definition 3.1.

For $x \in \mathbb{C}$ and $\alpha > -1$, the generalized q^2-cosine and q^2-sine are defined by:

$$\begin{aligned}\cos_\alpha(x;q^2) &:= \sum_{k=0}^{\infty}(-1)^k c_{k,\alpha}(x;q^2)\\ &= {}_1\phi_1\left(\begin{matrix}0\\ q^{2\alpha+2}\end{matrix}\,\middle|\, q^2;\; q^2x^2\right)\end{aligned} \tag{12}$$

and

$$\begin{aligned}\sin_\alpha(x;q^2) &:= \sum_{k=0}^{\infty}(-1)^k s_{k,\alpha}(x;q^2)\\ &= \frac{x}{1-q^{2\alpha+2}}\,{}_1\phi_1\left(\begin{matrix}0\\ q^{2\alpha+4}\end{matrix}\,\middle|\, q^2;\; q^2x^2\right)\end{aligned} \tag{13}$$

where we have put

$$c_{k,\alpha}(x;q^2) = \frac{q^{k(k+1)}\,x^{2k}}{(q^{2\alpha+2},q^2;q^2)_k}$$

and

$$s_{k,\alpha}(x;q^2) = \frac{q^{k(k+1)}\,x^{2k+1}}{(q^{2\alpha+2};q^2)_{k+1}(q^2;q^2)_k}.$$

By means of Eqs. (12) and (13), we define generalized q^2-analogue exponential function $e_\alpha(x;q^2)$ by

$$e_\alpha(x;q^2) = \cos_\alpha(-ix;q^2) + i\sin_\alpha(-ix;q^2). \tag{14}$$

Remark 3.1.

Compared with $\cos(x;q^2)$, $\sin(x;q^2)$ and $e(x;q^2)$, the generalized q^2-cosine and q^2-sine and exponential functions $\cos_\alpha(x;q^2)$, $\sin_\alpha(x;q^2)$ and $e_\alpha(x;q^2)$, respectively, involve two parameters "q" and "α". Clearly, $\cos(x;q^2)$, $\sin(x;q^2)$ and $e(x;q^2)$ can be considered as a special case of Eqs. (12)-(14), respectively. For $\alpha = -1/2$, we have:

$$\begin{aligned}\cos_{-\frac{1}{2}}(x;q^2) &= \cos(x;q^2),\\ \sin_{-\frac{1}{2}}(x;q^2) &= \sin(x;q^2),\\ e_{-\frac{1}{2}}(x;q^2) &= e(x;q^2).\end{aligned}$$

The relation between generalized q^2-cosine and q^2-sine functions and the classical hypergeometric functions is based on observations such as

$$\begin{aligned}&\lim_{q\to 1^-}\cos_\alpha((1-q)x;q^2)\\ &= {}_0F_1\left(\begin{matrix}-\\ \alpha+1\end{matrix}\,\middle|\, -\frac{x^2}{4}\right)\end{aligned}$$

and

$$\begin{aligned}&\lim_{q\to 1^-}\sin_\alpha((1-q)x;q^2) =\\ &\frac{x}{2(\alpha+1)}\,{}_0F_1\left(\begin{matrix}-\\ \alpha+2\end{matrix}\,\middle|\, -\frac{x^2}{4}\right).\end{aligned}$$

For $\alpha = -1/2$, we have:

$$\begin{aligned}\lim_{q\to 1^-}\cos_{-\frac{1}{2}}((1-q)x;q^2) &= \cos x,\\ \lim_{q\to 1^-}\sin_{-\frac{1}{2}}((1-q)x;q^2) &= \sin x,\\ \lim_{q\to 1^-} e_{-\frac{1}{2}}((1-q)x;q^2) &= e^x.\end{aligned}$$

The generalized q^2-exponential function $e_\alpha(x;q^2)$ is absolutely convergent for all x in the plane, $0 < q < 1$, since both generalized q^2-cosine and q^2-sine are absolutely convergent for all x in the plane, $0 < q < 1$.

We introduce generalized q-differential operator as

$$\begin{aligned}\partial_{q,\alpha}f(x) &= \frac{f(q^{-1}x)+f(-q^{-1}x)}{2(1-q)x}\\ &- \frac{q^{2\alpha+1}[f(x)+f(-x)]}{2(1-q)x}\\ &+ \frac{f(x)-f(-x)}{2(1-q)x}\\ &- \frac{q^{2\alpha+1}[f(qx)-f(-qx)]}{2(1-q)x},\; x\neq 0\end{aligned} \tag{15}$$

and

$$\partial_{q,\alpha}f(0) = \lim_{x\to 0}\partial_{q,\alpha}f(x) = [2\alpha+2]_q f'(0),$$

provided that $f'(0)$ exists.

We notice if f is differentiable at x,

$$\lim_{q\to 1^-}\partial_{q,\alpha}f(x) = f'(x).$$

Observe that, $\alpha = -1/2$ in Eq. (15) corresponds to the q^2-analogue differential operator [3], i.e., $\partial_{q,-1/2}f(x) = \partial_q f(x)$, and $\partial_{q,-1/2}f(0) = f'(0)$.

For all function f on $\mathbb{R}_{q,+}$, we have:

$$\partial_{q,\alpha} f(x) = \frac{f_e(q^{-1}x) - q^{2\alpha+1} f_e(x)}{(1-q)x} + \frac{f_o(x) - q^{2\alpha+1} f_o(qx)}{(1-q)x},$$

where f_e and f_o are respectively, even and odd parts of f. Since we have a realization of the generalized q-differential operator $\partial_{q,\alpha}$ in Eq. (16), we have:

$$\partial_{q,\alpha} \cos_\alpha((1-q)xt; q^2) = -t\sin_\alpha(1-q)xt; q^2) \quad (16)$$

$$\partial_{q,\alpha} \sin_\alpha(1-q)xt; q^2) = t\cos_\alpha((1-q)xt; q^2) \quad (17)$$

and

$$\partial_{q,\alpha}\ e_\alpha((1-q)xt;\ q^2) = te_\alpha((1-q)xt;\ q^2). \quad (18)$$

3.2 Generalized q^2-Fourier Transform

The goal is now to define a generalized q^2-deformed Fourier transform that formally tends to its classical analogue as $\alpha = -1/2$ and $q \to 1^-$.

For $1 \le p < \infty$, we denote by $L^p_{\alpha,q}(\mathbb{R}_{q,+})$ the space of complex-valued functions f on $\mathbb{R}_{q,+}$ such that:

$$\|f\|_{q,\alpha,p} = \left(\int_{-\infty}^{\infty} |f(x)|^p |x|^{2\alpha+1} d_q x\right)^{\frac{1}{p}} < \infty \quad (19)$$

and for $p = \infty$, we denote by $L^\infty_{\alpha,q}(\mathbb{R}_{q,+})$ the space of complex-valued functions f on $\mathbb{R}_{q,+}$ such that

$$\|f\|_{q,\alpha,\infty} = \sup_{x\in\mathbb{R}_{q,+}} \left\{|f(x)||x|^{2\alpha+1}\right\} < \infty.$$

The generalized q^2-Fourier transform will now be defined.

Definition 3.2.

Let f be a function in the space $L^1_{\alpha,q}(\mathbb{R}_{q,+})$. The generalized q^2-Fourier transform is defined by:

$$\hat{f}(x; q^2) := C_{\alpha,q} \times \int_{-\infty}^{\infty} f(t)\, e_\alpha(-i(1-q)tx; q^2) \times |t|^{2\alpha+1}\, d_q t,$$

where

$$C_{\alpha,q} = \frac{(1-q)^\alpha \left(q^{2\alpha+2}; q^2\right)_\infty}{2\left(q^2; q^2\right)_\infty}.$$

For $\alpha = -1/2$ and letting $q \uparrow 1$ subject to the condition,

$$\frac{Log(1-q)}{Log(q)} \in 2\mathbb{Z}, \quad (20)$$

gives, at least formally, the classical Fourier transform. In the remainder of this paper, we assume that the condition in Eq. (20) holds.

The following Lemma will be used to prove the inversion Theorem.

Lemma 3.1.

Let $f \in L^1_{\alpha,q}(d_q t)$. If $\int_{-\infty}^{\infty} f(x)|x|^{2\alpha+1} d_q x$ exists, with $(-1)^{2\alpha+1} = 1$, then:

- f odd implies $\int_{-\infty}^{\infty} f(x)|x|^{2\alpha+1} d_q x = 0$;
- f even implies $\int_{-\infty}^{\infty} f(x)\,|x|^{2\alpha+1} d_q x = 2\int_0^\infty f(x)\, x^{2\alpha+1} d_q x.$

Proposition 3.1.

For $f, g \in L^2_{\alpha,q}(\mathbb{R}_{q,+})$, the generalized q^2-cosine and q^2-sine transforms pair holds true.

$$g(q^n) = 2C_{\alpha,q} \times \int_0^\infty \left\{ \begin{array}{c} \cos_\alpha((1-q)xq^n; q^2) \\ \text{or} \\ \sin_\alpha((1-q)xq^n; q^2) \end{array} \right\} \times f(x) x^{2\alpha+1}\, d_q x \quad (21)$$

and

$$f(q^k) = 2C_{\alpha,q} \times \int_0^\infty \left\{ \begin{array}{c} \cos_\alpha((1-q)tq^k; q^2) \\ \text{or} \\ \sin_\alpha((1-q)tq^k; q^2) \end{array} \right\} \times g(t) t^{2\alpha+1}\, d_q t. \quad (22)$$

Proof. In order to prove the Proposition 3.1., we will start with the relation [4]:

$$\delta_{nm} = \sum_{k=-\infty}^{\infty} z^{k+n} \frac{(z^2;q)_\infty}{(q;q)_\infty} \times {}_1\phi_1\left(\begin{matrix} 0 \\ z^2 \end{matrix} \middle| q; q^{n+k+1} \right) \times z^{k+m} \frac{(z^2;q)_\infty}{(q;q)_\infty} {}_1\phi_1\left(\begin{matrix} 0 \\ z^2 \end{matrix} \middle| q; q^{m+k+1} \right) \tag{23}$$

where $|z| < 1$, $n, m \in \mathbb{Z}$.

Substituting q by q^2 and z by $q^{\alpha+1}$ into Eq. (23) yields

$$\delta_{nm} = \sum_{k=-\infty}^{\infty} \times q^{(\alpha+1)(k+n)} \frac{(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty} \cos_\alpha(q^{n+k};q^2) \times q^{(\alpha+1)(k+m)} \frac{(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty} \cos_\alpha(q^{m+k};q^2). \tag{24}$$

Rewrite the identity Eq. (24) as the transform pair

$$g(q^n) = \sum_{k=-\infty}^{\infty} q^{(n+k)(\alpha+1)} \frac{(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty} \times \cos_\alpha(q^{n+k};q^2) f(q^k) \tag{25}$$

and

$$f(q^k) = \sum_{n=-\infty}^{\infty} q^{(n+k)(\alpha+1)} \frac{(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty} \times \cos_\alpha(q^{n+k};q^2) g(q^n),$$

where f and g are $L^2_{\alpha,q}$ on the set $\{q^k, k \in \mathbb{Z}\}$ with respect to counting measure. Replacing in Eq. (25) $f(q^k)$, $g(q^n)$ by $q^{k(\alpha+1)}f(q^k)$, $q^{n(\alpha+1)}\, g(q^n)$, respectively, we obtain:

$$g(q^n) = \sum_{k=-\infty}^{\infty} q^{k(2\alpha+2)} \frac{(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty} \times \cos_\alpha(q^{n+k};q^2) f(q^k). \tag{26}$$

For such $q \in \{q^k, k \in \mathbb{Z}\}$, we can replace q^k, q^n in Eq. (26), by $(1-q)^{1/2}q^k$, $(1-q)^{1/2}q^n$. Then,

$$g((1-q)^{1/2}q^n) = \frac{(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty} \times \sum_{k=-\infty}^{\infty} q^{k(2\alpha+2)} \cos_\alpha((1-q)q^{n+k};q^2) \times (1-q)^{\alpha+1} f((1-q)^{1/2}q^k).$$

Next, replacing $f((1-q)^{1/2}q^k)$ and $g((1-q)^{1/2}q^n)$ by $f(q^k)$ and $g(q^n)$, we get:

$$g(q^n) = (1-q)^{\alpha+1} \frac{(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty} \times \sum_{k=-\infty}^{\infty} q^{k(2\alpha+2)} \cos_\alpha((1-q)q^{n+k};q^2) \times f(q^k). \tag{27}$$

With the q-integral notation in Eq. (9), the relation in Eq. (27) is equivalent to

$$g(t) = 2C_{\alpha,q} \int_0^\infty \cos_\alpha((1-q)tx;q^2) \times f(x) x^{2\alpha+1}\, d_qx.$$

The proof of Eq. (21) is achieved. Similarly, we can prove Eq. (22).

Remark 3.2.

For $\alpha = -1/2$ and $q \uparrow 1$ in assertions of Eqs. (21) and (22), we get the classical Fourier pair:

$g(\lambda) =$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \left\{ \begin{matrix} \cos(x\lambda) \\ \text{or} \\ \sin(x\lambda) \end{matrix} \right\} f(x)\, dx,$$

$f(x) =$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \left\{ \begin{matrix} \cos(x\lambda) \\ \text{or} \\ \sin(x\lambda) \end{matrix} \right\} g(\lambda)\, d\lambda.$$

Lemma 3.2.

For $f, g \in L^2_{\alpha,q}(\mathbb{R}_{q,+})$, the transformations $f \longmapsto g$ and $g \longmapsto f$ of Eqs. (21) and (22) establish isometry of Hilbert spaces:

$$\sum_{k=-\infty}^{+\infty} q^{k(2\alpha+2)}|f(q^k)|^2 = \sum_{n=-\infty}^{+\infty} q^{n(2\alpha+2)}|g(q^n)|^2. \tag{28}$$

Let us now turn to the L^2 theory of the generalized q^2-Fourier transform. Since the generalized q^2-Fourier transform is defined and bounded on $(L^1_{\alpha,q} \cap L^2_{\alpha,q})(d_qt)$ (dense in $L^2_{\alpha,q}(d_qt)$ for the functions with finite support), it defines a bounded extension to all of $L^2_{\alpha,q}(d_qt)$.

We can use Lemma 3.1. and Proposition 3.1. to prove the following theorem.

Theorem 3.1.

$f \in (L^1_{\alpha,q} \cap L^2_{\alpha,q})(d_qt)$ implies

$f(x)$ =

$$C_{\alpha,q}\int_{-\infty}^{\infty} \hat{f}(t;q^2)\, e_\alpha(i(1-q)tx;q^2) \times |t|^{2\alpha+1}\, d_qt,\ \forall x \in \mathbb{R}_{q,+}. \tag{29}$$

Theorem 3.2.

Let f be the functions with finite support in $L^2_{\alpha,q}(d_qt)$. $f \in L^2_{\alpha,q}(d_qt)$ implies

$$\|f\|_{q,\alpha,2} = \|\hat{f}(\cdot;q^2)\|_{q,\alpha,2}. \tag{30}$$

4. Conclusions

In our present investigation, we have constructed a pair of potentially generalized q^2-cosine, q^2-sine and q^2-exponential functions. We then have successfully used $e_\alpha(x;\ q^2)$ to define and investigate generalized q^2-Fourier transform. In particular, we have established q-analogues of inversion and Plancherel theorems.

References

[1] Jackson, F. H. 1910. "On a q-Definite Integrals." *Quarterly Journal of Pure and Applied Mathematics* 41: 193-203.

[2] Jackson, F. H. 1908. "On a q-Functions and Certain Difference Operator." *Transactions of the Royal Society of London* 46: 253-81.

[3] Rubin, R. L. 1997. "A q^2-Analogue Operator for q^2-Analogue Fourier Analysis." *J. Math. Anal. App.* 212: 571-82.

[4] Koornwinder, T. H., and Swarttouw, R. F. 1992. "On q-Analogues of the Fourier and Hankel Transforms." *Trans. Amer. Math. Soc.* 333: 445-61.

[5] Exton, H. 1983. *Basic Hypergeometric Functions and Applications*. Chichester: Ellis Horwood.

[6] Hahn, W. 1953. "Die mechanishce Deutungeiner geometrischen Differenzengleichung." *Z. Angew. Math. Mech.* 33: 270-2.

[7] Vaksman, L. L., and Korogodskii, L. I. 1989. "An Algebra of Bounded Functions on the Quantum Group of the Motions of the Plane, and q-Analogues of Bessel Functions." *Soviet Math. Dokl.* 39: 173-7.

[8] Kalnins, E. G., Miller, W., and Mukherjee, S. 1994. "Models of q-Algebra Representations: The Group of Plane Motions." *SIAM J. Math. Anal.* 25: 513-27.

[9] Koelink, H. T., and Swarttouw, R. F. 1994. "On the Zeroes of the Hahn-Exton q-Bessel Function and Associated q-Lommel Polynomials." *J. Math. Anal. Appl.* 186: 690-710.

[10] Koelink, H. T. 1994. "The Quantum Group of Plane Motions and the Hahn-Exton q-Bessel Function." *Duke Math. J.* 76: 483-508.

[11] Swarttouw, R. F., and Meijer, H. G. 1994. "A q-Analogue of the Wronskian and a Second Solution of the Hahn-Exton q-Bessel Difference Equation." *Proc. Amer. Math. Soc.* 120: 855-64.

[12] Gasper, G., and Rahman, M. 1990. *Basic Hypergeometric Series, Encyclopedia of Mathematics and Its Application*. Vol, 35, Cambridge, UK: Cambridge Univ. Press.

[13] Koekoek, R., and Swarttouw, R. 1998. *The Askey-Scheme of Hypergeometric Orthogonal Polynomials and Its q-Analogue*. Delft Report 98-17, the Netherlands.

[14] Andrews, G. E., Askey, R., and Roy, R. 1999. *Special Functions, Encyclopedia of Mathematics and Its Applications Sciences*. Vol. 71, Cambridge: Cambridge University Press.

[15] Jazmati, M., Mezlini, K., and Bettaibi, N. 2014. "Generalized q-Hermite Polynomials and the q-Dunkl Heat Equation." *Bull. Math. Anal. Appl.* 6 (4): 16-43.

[16] Kac, V. G., and Cheung, P. 2002. *Quantum Calculus, Universitext*. New York: Springer-Verlag.