

Solubility Existence of Inverse Eigenvalue Problem for a Class of Singular Hermitian Matrices

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Abstract: In this article, we discuss singular Hermitian matrices of rank greater or equal to four for an inverse eigenvalue problem. Specifically, we look into how to generate n by n singular Hermitian matrices of ranks four and five from a prescribed spectrum. Numerical examples are presented in each case to illustrate these scenarios. It was established that given a prescribed spectral datum and it multiplies, then the solubility of the inverse eigenvalue problem for n by n singular Hermitian matrices of rank r exists.

Key words: Singular hermitian matrices, inverse eigenvalue problem, rank of a matrix.

1. Introduction

A matrix A is said to have rank r if it contains at least one r -rowed square submatrix with a nonzero determinant, whereas the determinant of any square submatrix having $r + 1$ or more rows, possibly contained in A , is zero. A is said to have rank zero if all its elements are zero. Any n -rowed square matrix A has rank $r < n$ if and only if $|A| = 0$. A is called a singular matrix in this case. The matrix has rank $r = n$ if and only if $|A| \neq 0$ and is then called a nonsingular matrix [1].

An inverse eigenvalue problem deals with the reconstruction of a matrix from its spectrum. The aim of an inverse eigenvalue problem is to construct a matrix that maintains a certain specific structure as well as the given spectral property. The fundamental questions associated with any inverse eigenvalue problem are the issue of solvability and computability.

Effort in solvability has been made to determine a necessary or a sufficient condition under which an inverse eigenvalue has a solution. On the other hand, the concern of computability has been used to develop

a procedure by which a matrix can be numerically constructed. These two fundamental questions are challenging and difficult. Inverse eigenvalue problem arises in a broad variety of applications. It includes but is not limited to circuit theory, particle physics, structure analysis, exploration and remote sensing, system identification, principal component analysis and control design.

Research on inverse eigenvalue problem has been intensive, ranging from algebraic theory to engineering application, but the results are widely spread even within the same field of discipline. Only a handful of the problems discussed have been solved, despite the great effort found in this area of research work. Recently the solution to the inverse eigenvalue problem for singular symmetric matrices of rank, $r = 1$ and singular Hermitian matrices of rank, $r = 2$ and $r = 3$ have been studied respectively in Refs. [2, 3]. In this research, solution of the inverse eigenvalue problem for singular Hermitian matrices of rank, $r = 4$ and $r = 5$ is considered.

This research is organized as follows: Review of how to reconstruct singular Hermitian matrices of rank 2 and 3 from its prescribed spectrum is looked at in Section 2. We present the new result on the solution to inverse eigenvalue problem for singular Hermitian

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matrices of rank 4 and 5 in Section 3. Conclusion and recommendation are given in the last section.

2. Preliminaries

In this section, we take a look at a review of some previous results obtained by other authors in respect of solution to an inverse eigenvalue problem for singular Hermitian matrices of rank $r \leq 3$ and some necessary proposition, lemmas, corollary and theorems associated with this research work.

We begin with 2×2 singular Hermitian matrices of rank $r = 2$. Singularity is obtained by multiplying the first row by prescribed scalars. We denote an $n \times n$ singular Hermitian matrix of rank r by $A_{(n,r)}$. In this case, we write $R_{i+1} = kR_i$ to denote the i th row is k times the first row, where $k \in R$. We also denote the spectrum of A_n by $\Lambda_n = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$. If the rank of A is r , then we assume $\lambda_i \neq 0$ for $i = 1, \dots, r$ but $\lambda_i = 0, i = r + 1, \dots, n$.

Lemma 2.1. Let A be a non-traceless, symmetric matrix of rank r with non-vanishing elements. Then there exists a one-to-one correspondence between the elements of A and its distinct non-zero eigenvalues if and only if $r = 1$ [4].

Proof:

Let the given Hermitian matrix be of rank r , then, clearly, the number of independent elements is $r(r + 1)/2$. Thus a one-to-one correspondence will exist between the elements of the matrix and its distinct nonzero eigenvalues if and only if $r(r + 1)/2 = r$, i.e. if and only if $r = 1$.

Corollary 2.1. The inverse eigenvalue problem has a unique solution for singular symmetric matrices of rank one with prescribed linear dependence relation.

Proposition 2.1. If the row dependence relations for a Hermitian matrix of rank 1 are specified as follows:

$$R_i = k_{i-1}R_1, i = 2, \dots, n-1$$

where R_i is the i th row and each k_i is a non-zero scalar, then the matrix can be generated from its non-zero eigenvalue λ .

Proof: See proof in Ref. [5].

Proposition 2.2. The trace of an $n \times n$ square matrix A is the sum of the eigenvalues.

Proof: See proof in Ref. [2].

Proposition 2.3. If a square matrix A has one row (column) as a scalar multiple of another row (column), then A is a singular matrix and $\det A = 0$.

Lemma 2.2. Entries of singular Hermitian matrix $A = [a_{ij}]$ of rank $r = 1$ with nonzero eigenvalue $\lambda \in R$ and which is such that $R_i = k_{i-1}R_{i-1}$ where $k_i \in C, i = 2, \dots, n$, and R_i is the rows of A and the row can be generated in the form:

$$a_{11} = \frac{\lambda}{1 + |k_1|^2 + |k_1|^2|k_2|^2 + \dots + |k_1|^2 \dots |k_{n-1}|^2}$$

and

$$a_{ij} = a_{11} \left(\prod_{s=0}^{i-1} k_s \right)$$

Theorem 2.1. Given the spectrum and the row multipliers for $k_i, i = 1, \dots, n-2$, the inverse eigenvalue problem for $n \times n$ singular Hermitian matrix of rank 2 is solvable.

Proof: See proof in Ref. [2].

Theorem 2.2. The inverse eigenvalue problem for an $n \times n$ singular Hermitian matrix of rank r is solvable provided that $n-r$ arbitrary parameters are prescribed.

We now use numerical examples to illustrate singular Hermitian matrices of size $2 \leq n \leq 4$ and of ranks 2 and 3.

Case 1: Given $n = 3, r = 2$, the singular Hermitian matrix $A_{(3,2)}$ is of the form:

$$A_{(3,2)} = \begin{pmatrix} a_{11} & \bar{k}a_{11} & \bar{a}_{13} \\ ka_{11} & |k|^2 a_{11} & k\bar{a}_{13} \\ a_{13} & \bar{k}a_{13} & a_{33} \end{pmatrix}$$

$\text{tr}(A_{(3,2)}) = \lambda_1 + \lambda_2 = a_{11}(1 + |k|^2) + a_{33}$. Thus $a_{11} = \lambda_1/(1 + |k|^2)$ and $\lambda_2 = a_{33}$. a_{13} is a free variable.

Suppose $\lambda_1 = 1, \lambda_2 = 2, k = 3i$ and $a_{13} = 2 + i$, the following singular Hermitian matrix $A_{(3,2)}$ is obtained as follows:

$$A_{(3,2)} = \begin{pmatrix} \frac{1}{10} & \frac{-3i}{10} & 2-i \\ \frac{3i}{10} & \frac{9}{10} & 3+6i \\ 2+i & 3-6i & 2 \end{pmatrix}$$

Case 2: If $n = 4$, $r = 2$, then $A_{(4,2)}$ is of the form:

$$A_{(4,2)} =$$

$$\begin{pmatrix} a_{11} & \bar{k}_1 a_{11} & \bar{k}_1 \bar{k}_2 & \bar{a}_{14} \\ k_1 a_{11} & |k_1|^2 a_{11} & |k_1|^2 \bar{k}_2 a_{11} & k_1 \bar{a}_{14} \\ k_1 k_2 a_{11} & |k_1|^2 k_2 a_{11} & |k_1|^2 |k_2|^2 a_{11} & k_1 k_2 \bar{a}_{14} \\ a_{14} & \bar{k}_1 a_{14} & \bar{k}_1 \bar{k}_2 a_{14} & a_{44} \end{pmatrix}$$

$$\text{tr}(A_{(4,2)}) = \lambda_1 + \lambda_2 = a_{11}(1 + |k_1|^2 + |k_1|^2 |k_2|^2) + a_{44}$$

This implies that:

$a_{11} = \lambda_1/1 + |k_1|^2 + |k_1|^2 |k_2|^2$ and $\lambda_2 = a_{44}$. a_{14} is a free variable.

If $\lambda_1 = 3$, $\lambda_2 = 5$, $k_1 = 2i$, $k_2 = 1 + 2i$ and $a_{14} = 1 + i$, a singular Hermitian matrix is generated below:

$$A_{(4,2)} =$$

$$\begin{pmatrix} \frac{3}{25} & \frac{-6i}{25} & \frac{-12-6i}{25} & 1-i \\ \frac{6i}{25} & \frac{12}{25} & \frac{12-24i}{25} & 2+2i \\ \frac{-12+6i}{25} & \frac{12+24i}{25} & \frac{60}{25} & -2+6i \\ 1+i & 2-2i & -2-6i & 5 \end{pmatrix}$$

Case 3: Here the inverse eigenvalue problem for 4×4 singular Hermitian matrices of rank 3 is considered.

Thus $A_{(4,3)}$ is of the form:

$$A_{(4,3)} =$$

$$\begin{pmatrix} a_{11} & \bar{k} a_{11} & \bar{a}_{13} & \bar{a}_{14} \\ k a_{11} & |k|^2 a_{11} & k \bar{a}_{13} & k \bar{a}_{14} \\ a_{13} & \bar{k} a_{13} & a_{33} & \bar{a}_{34} \\ a_{14} & k a_{14} & a_{34} & a_{44} \end{pmatrix}$$

$$\text{tr}(A_{(4,3)}) = \lambda_1 + \lambda_2 + \lambda_3 = a_{11}(1 + |k|^2) + a_{33} + a_{44}.$$

A singular Hermitian matrix $A_{(4,3)}$ is obtained if we let $\lambda_1 = 2$, $\lambda_2 = -1$, $\lambda_3 = 5$, $k = -i$, $a_{13} = 2 + i$, $a_{14} = 2i$

and $a_{34} = 1-3i$:

$$A_{(4,3)} =$$

$$\begin{pmatrix} 1 & i & 2-i & -2i \\ -i & 1 & -1-2i & -2 \\ 2+i & -1+2i & -1 & 1+3i \\ 2i & -2 & 1-3i & 5 \end{pmatrix}$$

Case 4: Lastly we present 5×5 singular Hermitian matrices of rank 3. Then $A_{(5,3)}$ will be of the form:

$$A_{(5,3)} =$$

$$\begin{pmatrix} a_{11} & \bar{k}_1 a_{11} & \bar{k}_1 \bar{k}_2 a_{11} & \bar{a}_{14} & \bar{a}_{15} \\ k_1 a_{11} & |k_1|^2 a_{11} & |k_1|^2 \bar{k}_2 a_{11} & k_1 \bar{a}_{14} & k_1 \bar{a}_{15} \\ k_1 k_2 a_{11} & |k_1|^2 k_2 a_{11} & |k_1|^2 |k_2|^2 a_{11} & k_1 k_2 \bar{a}_{14} & k_1 k_2 \bar{a}_{15} \\ a_{14} & \bar{k}_1 a_{14} & \bar{k}_1 \bar{k}_2 a_{14} & a_{44} & \bar{a}_{45} \\ a_{15} & \bar{k}_1 a_{15} & \bar{k}_1 \bar{k}_2 a_{15} & a_{45} & a_{55} \end{pmatrix}$$

$\text{tr}(A_{(5,3)}) = \lambda_1 + \lambda_2 + \lambda_3 = a_{11}(1 + |k_1|^2 + |k_1|^2 |k_2|^2) + a_{44} + a_{55}$. This implies that $a_{11} = \lambda_1/1 + |k_1|^2 + |k_1|^2 |k_2|^2$, $\lambda_2 = a_{44}$ and $\lambda_3 = a_{55}$. a_{14} , a_{15} and a_{55} are the free variables.

Now given $\lambda_1 = 13$, $\lambda_2 = -3$, $\lambda_3 = 5$, $k_1 = 2i$, $k_2 = 1 + i$, $a_{14} = 4$, $a_{15} = i$ and $a_{45} = 3-i$, we obtain the 5×5 singular Hermitian matrix below:

$$A_{(5,3)} =$$

$$\begin{pmatrix} 1 & -2i & -2(1+i) & 4 & -i \\ 2i & 4 & 4(1-i) & 8i & 2 \\ -2(1-i) & 4(1+i) & 8 & -8(1-i) & 2(1+i) \\ 4 & -8 & -8(1+i) & -3 & 3+i \\ i & 2 & 2(1-i) & 3-i & 5 \end{pmatrix}$$

3. Main Result

At this stage of our study, we focus on inverse eigenvalue problem for $n \times n$ singular Hermitian matrices of 4th and 5th rank, where rank $r < n$.

Singular Hermitian matrices of size $n = 5$ and of rank $r = 4$ denoted by $A_{(5,4)}$ are of the form:

$$A_{(5,4)} = \begin{pmatrix} a_{11} & \bar{k}a_{11} & \bar{a}_{13} & \bar{a}_{14} & \bar{a}_{15} \\ ka_{11} & |k|^2 a_{11} & k\bar{a}_{13} & k\bar{a}_{14} & k\bar{a}_{15} \\ a_{13} & \bar{k}a_{13} & a_{33} & \bar{a}_{34} & \bar{a}_{35} \\ a_{14} & \bar{k}a_{14} & a_{34} & a_{44} & \bar{a}_{45} \\ a_{15} & \bar{k}a_{15} & a_{35} & a_{45} & a_{55} \end{pmatrix}$$

The free variables are a_{13} , a_{14} , a_{15} , a_{34} , a_{35} and a_{45}
 $\text{tr}A_{(5,4)} = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = a_{11}(1 + |k|^2) + a_{33} + a_{44} + a_{55}$.
 Thus $a_{11} = \lambda_1/(1 + |k|^2)$, $\lambda_2 = a_{33}$, $\lambda_3 = a_{44}$ and $\lambda_4 = a_{55}$.

For instance, given $\lambda_1 = 5$, $\lambda_2 = 2$, $\lambda_3 = -1$, $\lambda_4 = 3$, $k = -2i$, $a_{13} = 1 + i$, $a_{14} = i$, $a_{15} = 2i$, $a_{34} = 1 - 2i$, $a_{35} = 1 + 2i$, and $a_{45} = 1 + 3i$, a singular Hermitian matrix $A_{(5,4)}$ is generated below:

$$A_{(5,4)} = \begin{pmatrix} 1 & 2i & 1-i & -i & -2i \\ -2i & 4 & 1-2i & 2 & 4 \\ 1+i & -2+2i & 2 & 1+2i & 1-2i \\ i & -2 & 1-2i & -1 & 1-3i \\ 2i & -4 & 1+2i & 1+3i & 3 \end{pmatrix}$$

Next we consider singular Hermitian matrices $A_{(6,4)}$ which is of the form below:

$$A_{(6,4)} = \begin{pmatrix} a_{11} & \bar{k}_1 a_{11} & \bar{k}_1 \bar{k}_2 a_{11} & \bar{a}_{14} & \bar{a}_{15} & \bar{a}_{16} \\ k_1 a_{11} & |k_1|^2 a_{11} & |k_1|^2 \bar{k}_2 a_{11} & k_1 \bar{a}_{14} & k_1 \bar{a}_{15} & k_1 \bar{a}_{16} \\ k_1 k_2 a_{11} & |k_1|^2 k_2 a_{11} & |k_1|^2 |k_2|^2 a_{11} & k_1 k_2 \bar{a}_{14} & k_1 k_2 \bar{a}_{15} & k_1 k_2 \bar{a}_{16} \\ a_{14} & \bar{k}_1 a_{14} & \bar{k}_1 \bar{k}_2 a_{14} & a_{44} & \bar{a}_{45} & \bar{a}_{46} \\ a_{15} & \bar{k}_1 a_{15} & \bar{k}_1 \bar{k}_2 a_{15} & a_{45} & a_{55} & \bar{a}_{56} \\ a_{16} & \bar{k}_1 a_{16} & \bar{k}_1 \bar{k}_2 a_{16} & a_{46} & a_{56} & a_{66} \end{pmatrix}$$

The free variables in this case are a_{14} , a_{15} , a_{16} , a_{45} , a_{46} and a_{56} .

$\text{tr}(A_{(6,4)}) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = a_{11}(1 + |k_1|^2 + |k_1|^2 |k_2|^2) + a_{44} + a_{55} + a_{66}$. This implies that $a_{11} = \lambda_1/(1 + |k_1|^2 + |k_1|^2 |k_2|^2)$, $\lambda_2 = a_{44}$, $\lambda_3 = a_{55}$ and $\lambda_4 = a_{66}$.

For example, when $k_1 = i$, $k_2 = 2i$, $\lambda_1 = 12$, $\lambda_2 = 3$, $\lambda_3 = 4$, $\lambda_4 = 5$, $a_{14} = 1 + i$, $a_{15} = 3i$, $a_{16} = -2i$, $a_{45} = -i$, $a_{46} = 2 + i$ and $a_{56} = 3 + i$ is given, a singular Hermitian $A_{(6,4)}$ is formed below:

$$A_{(6,4)} = \begin{pmatrix} 2 & -2i & -4 & 1-i & -3i & 2i \\ 2i & 2 & -4i & 1+i & 3 & -2 \\ -4 & 4i & 8 & -2(1-i) & 6i & -4i \\ 1+i & 1-i & -2(1+i) & 3 & i & 2-i \\ 3i & 3 & -6i & -i & 4 & 3-i \\ -2i & -2 & 4i & 2+i & 3+i & 5 \end{pmatrix}$$

Finally we look at singular Hermitian matrices $A_{(6,5)}$ which is of the form:

$$A_{(6,5)} = \begin{pmatrix} a_{11} & \bar{k}a_{11} & \bar{a}_{13} & \bar{a}_{14} & \bar{a}_{15} & \bar{a}_{16} \\ ka_{11} & |k|^2 a_{11} & k\bar{a}_{13} & k\bar{a}_{14} & k\bar{a}_{15} & k\bar{a}_{16} \\ a_{13} & \bar{k}a_{13} & a_{33} & \bar{a}_{34} & \bar{a}_{35} & \bar{a}_{36} \\ a_{14} & \bar{k}a_{14} & a_{34} & a_{44} & \bar{a}_{45} & \bar{a}_{46} \\ a_{15} & \bar{k}a_{15} & a_{35} & a_{45} & a_{55} & \bar{a}_{56} \\ a_{16} & \bar{k}a_{16} & a_{36} & a_{46} & a_{56} & a_{66} \end{pmatrix}$$

In this case the free variables are a_{13} , a_{14} , a_{15} , a_{16} , a_{34} , a_{35} , a_{36} , a_{45} , a_{46} and a_{56} .

$\text{tr}(A_{(6,5)}) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = a_{11}(1 + |k|^2) + a_{33} + a_{44} + a_{55} + a_{66}$. Thus $a_{11} = \lambda_1/(1 + |k|^2)$, $\lambda_2 = a_{33}$, $\lambda_3 = a_{44}$, $\lambda_4 = a_{55}$, and $\lambda_5 = a_{66}$.

Suppose $k = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\lambda_4 = 4$, and $\lambda_5 = 5$, a_{13}

$= i, a_{14} = -2i, a_{15} = 3i, a_{16} = 4i, a_{34} = 1 + i, a_{35} = 2 + i, a_{36} = 3 + i, a_{45} = 4 + i, a_{46} = 5 + i$ and $a_{56} = 1 - i$, then a singular Hermitian matrix $A_{(6,5)}$ is generated as follows:

Proposition 3.1 The solution of the inverse eigenvalue problem for $A_{[n,r]}$ leads to the solution of an r th degree polynomial equation in a_{11} of the form [2]:

$$A_{(6,5)} = \begin{pmatrix} 1 & -2i & -i & 2i & -3i & -4i \\ 2i & 4 & 2 & -4 & 6 & 8 \\ i & 2 & 2 & 1-i & 2-i & 3-i \\ -2i & -4 & 1+i & 3 & 4-i & 5-i \\ 3i & 6 & 2+i & 4+i & 4 & 1+i \\ 4i & 8 & 3+i & 5+i & 1-i & 5 \end{pmatrix}$$

$$a_{11}^r (1 + |k_1|^2 + \dots + |k_{n-1}|^2)^r - \left(\sum_{i=1}^r \lambda_i \right) (1 + |k_1|^2 + \dots + |k_{n-r}|^2) a_{11}^{r-1} + \sum_{k=1}^r \left(\prod_{i=k}^{k+1} \lambda_i \right) (1 + |k_1|^2 + \dots + |k_{n-r}|^2) a_{11}^{r-2} - \sum_{k=1}^{k+2} \left(\prod_{i=k}^{k+2} \lambda_i \right) (1 + |k_1|^2 + \dots + |k_{n-r}|^2) a_{11}^{r-3} + \dots - \left(\prod_{i=1}^r \lambda_i \right) = 0$$

4. Conclusion

It has been established that infinitely many solutions exist for $n \times n$ singular Hermitian matrices of rank r , where $3 < r < n$, when given the eigenvalues and some specified parameters. Numerical examples were provided to illustrate these results.

We recommend the use of a mathematical software package for the numerical computation of an $n \times n$ singular Hermitian matrix of rank $r \geq 6$.

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