Journal of Mathematics and System Science 9 (2019) 86-89
doi: 10.17265/2159-5291/2019.03.002

Principally-Injective Leavitt Path Algebras over Arbitrary Graphs

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Abstract: A ring R is called right *principally-injective* if every R-homomorphism f: $aR \rightarrow R$, $a \in R$, extends to R, or equivalently, if every system of equations $xa = b$ (a, $b \in R$) is solvable in R. In this paper we show that for any arbitrary graph E and for a field K, principally-injective conditions for the Leavitt path algebra $L_K(E)$ are equivalent to that graph E being acyclic. We also show that the principally-injective Leavitt path algebras are precisely the von Neumann regular Leavitt path algebras.

Key words: Leavitt path algebras, von Neumann regular rings, principally-injective rings, arbitrary graph.
2010 Mathematics Subject Classification. 16D50, 16D60.

1. Introduction

All the rings that we consider here are assumed to be associative with local units (such as the Leavitt path algebras).

One of the fascinating directions to study in Leavitt path algebras is the characterization of the ring-theoretic properties of a Leavitt path algebra $L_K(E)$ in terms of the graph-theoretic properties of the graph E (see chapter 4 [1]). This motivates us to study Principally-injective Leavitt path algebras.

Recall that a ring R is *locally unital* if for each finite set F of elements of R, there is an idempotent u (i.e. $u^2 = u \in R$) such that $ua = au = a$ for all $a \in F$. The set of all such idempotents u is said to be a set of local units. A ring R is said to be (von Neumann) regular if each $a \in$ R satisfies a $\in aRa$. The von Neumann regular Leavitt path algebras $L_K(E)$ of arbitrary graphs E over a field K were characterized in Ref. [2] in terms of the graphical properties of E, namely, the graphs E must have no cycles.

For a subset X of a ring R (not necessarily unital), the set $r_R(X) = \{t \in R\text{: } xt = 0 \ \forall x \in X\}$ is right annihilator of X. The left annihilator $l_R(X)$ is also defined in a similar fashion for $X \subseteq R$. It is straightforward to check that $r_R(X)$ is a right and $l_R(X)$ is a left ideal of R. A ring R is called right *principally injective* (*P-injective*) if every R-homomorphism
f: $aR \rightarrow R$, $a \in R$, extends to g: $R \rightarrow R$ or equivalently (see Lemma 3.1), if every system of equations $xa = b$ (a, $b \in R$) has a solution x in R. Thus every right self-injective ring is right P-injective. We shall see in Lemma 3.2 that the following are equivalent for a (locally unital) ring R (i) R is right P-injective (ii) $l_R r_R(a) = Ra$ for all $a \in R$. We note in Lemma 3.4 that every (locally unital) regular ring R is both right and left P-injective. As consequences we will prove in Theorem 3.9 that every regular Leavitt path algebra $L_K(E)$ is regular if and only if $L_K(E)$ is P-injective if and only if the graph E contains no cycle.

For the other definitions in this note, we refer to Refs. [4-5].

2. Preliminaries

We recall the fundamental terminology for our note which can be found in the text [1]. For the sake of completeness, we shall outline some of the concepts and results that we will be using. A (directed) graph $E = (E^0; E^1; r; s)$ consists of two sets E^0 and E^1 together with maps r, s: $E^1 \rightarrow E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. For each $e \in E^1$,

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$r(e)$ is the range of e and $s(e)$ is the source of e. If $r(e) = v$ and $s(e) = w$, then we say that v emits e and that w receives e. A vertex v is called a *sink* if it emits no edges and a vertex v is called a *regular* vertex if it emits a non-empty finite set of edges. An *infinite emitter* is a vertex which emits infinitely many edges. For each $e \in E^1$, we call e^* a ghost edge. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. A path μ of length $n \gneqq 0$ is a finite sequence of edges $\mu = e_1e_2 \dots e_n$ with $r(e_i) = s(e_i+1)$ for all $i = 1, \dots, n$-1. In this case $\mu^* = e_n^* \dots e_2^*e_1^*$ is the corresponding ghost path. A vertex is considered a path of length 0. For a vertex v, we define $v^* = v$. The set of all vertices on the path μ is denoted by μ^0. The set of all paths in E is denoted by Path(E). A path $\mu = e_1e_2 \dots e_n$ in E is closed if $r(e_n) = s(e_n)$, in which case μ is said to be based at the vertex $s(e_1)$. A closed path μ as above is called *simple* provided it does not pass through its base more than once, i.e., $s(e_i) \neq s(e_1)$ for all $i = 2, \dots, n$. The closed path μ is called a *cycle* if it does not pass through any of its vertices twice, that is, if $s(e_i) \neq s(e_j)$ for every $i \neq j$.

Given an arbitrary graph E and a field K, the Leavitt path algebra $L_K(E)$ is defined to be the K-algebra generated by a set $\{v : v \in E^0\}$ of pair-wise orthogonal idempotents together with a set of variables $\{e, e^*: e \in E^1\}$ which satisfy the following conditions:

(1) $s(e)e = e = er(e)$ for all $e \in E^1$.

(2) $r(e)e^* = e^* = e^*s(e)$ for all $e \in E^1$.

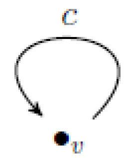
(3) (The CK-1 relations) For all $e, f \in E^1$, $e^*e = r(e)$ and $e^*f = 0$ if $e \neq f$.

(4) (The CK-2 relations) For every regular vertex $v \in E^0$, $v = \sum_{e\in E1, s(e)=v} ee^*$.

A useful observation is that every element a of $L_K(E)$ can be written in the form a = $\sum_{i=1}^{n} \kappa i \alpha i \beta i^*$, where $\kappa_i \in K$, α_i, β_i are paths in E and n is a suitable integer.

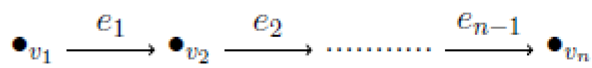
We mention two basic examples:

(1) If E is the graph having one vertex and a single loop:



then $L_K(E) \cong K[x, x^{-1}]$, the Laurent polynomial K-algebra via $v \mapsto 1$, $c \mapsto x$ and $c^* \mapsto x^{-1}$.

(2) If E is the oriented n-line graph having n vertices and n-1 edges:



then $L_K(E) \cong M_n(K)$, via $v_i \mapsto f_{i,i}$, $e_i \mapsto f_{i,i+1}$ and $e_i^* \mapsto f_{i+1,i}$, where $\{f_{i,j}: 1 \leq i, j \leq n\}$ denotes the standard matrix units in $M_n(K)$.

3. Results

We start with the following observation.

Lemma 3.1. (cf. Proposition 3.17 [4]). The following conditions are equivalent for a locally unital ring R.

(1) Every R-homomorphism f: $aR \to R$, $a \in R$, extends to g: $R \to R$

(2) Every system of equations $xa = b$ ($a, b \in R$) in R has a solution x in R

Proof. (1) $\Longrightarrow$ (2). Consider a system of equations $xa = b$ in R with $a, b \in R$. Define f: $aR \to R$ by $f(ar) = br$, $\forall r \in R$. Then f is well defined as $ar = ar'$ implying that a(r-r')= 0, that is (r-r') $\in r_R(a) \subseteq r_R(b)$. So, $f(ar) = br = br' = f(ar')$. Clearly, f is a right R-homomorphism. By (1), there exists g: $R \to R$ such that $g(a) = f(a)$. Then $b = f(a) = g(a) = g(ua) = g(u)a$, where u is the local unit for a. Hence $x = g(u)$ is the required solution in R.

(2) $\Longrightarrow$ (1). Let f: $aR \to R$, $a \in R$, be R-linear. Then $f(a) = b$ for some $b \in R$. By (2), $xa = b$ is solvable in R. Write $x = c$ in R such that $ca = b$. Define g: $R \to R$ by $g(r) = cr$, $\forall r \in R$. Then g is the required extension of f on R.

We shall need the following lemmas.

Lemma 3.2. (cf. Lemma 5.1 [5]). The following conditions are equivalent for a locally unital ring R:

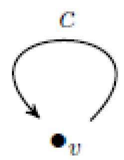
(i) R is right P-injective.

(ii) $l_R r_R(a) = Ra$ for all a ∈ R.

Proof. (i) ⟹ (ii). For any $z \in r_R(a)$, $az = 0$. This implies that $a \in l_R(z)$, $\forall z \in r_R(a)$, yielding $Ra \subseteq l_R r_R(a)$. Let $x \in l_R r_R(a)$. Then, $r_R(a) \subseteq r_R(x)$ .Define f: $aR \to R$ by $f(at) = xt$. This is well-defined as $at = at'$ impling that $a(t\text{-}t') = 0$, that is $(t\text{-}t') \in r_R(a) \subseteq r_R(x)$. So, $f(at) = xt = xt' = f(at')$. By (i), there exists
g: $R \to R$ such that $f(a) = g(a) = g(ua) = g(u)a \in Ra$, where u is the local unit for a. Hence $l_R r_R(a) = Ra$.

(ii) ⟹ (i) Let f: $aR \to R$, $a \in R$, be R-linear. Then $f(a) = d$, for some $d \in R$. We show that $d \in Ra$. Take $x \in r_R(a)$. Then $0 = f(ax) = f(a)x = dx$, so, $r_R(a) \subseteq r_R(d)$. This implies that $l_R r_R(d) = Ra \subseteq l_R r_R(a)$. So, $d \in l_R r_R(a) = Ra$, therefore, $d \in Ra$. Hence $f(a) = d = ca$ for some $c \in R$. Define g: $R \to R$ by $g(x) = cx$. Then g is the required extension of f on R.

Before deriving the next lemma, we insert a remark here.

Remark 3.3. For the graph E = 

the corresponding Leavitt path algebra $R = L_K(E)$ is not right (left) P-injective can be seen as follows: $c^* \in l_R r_R(v\text{-}c)$ but $c^* \notin R(v\text{-}c)$. Thus
$l_R r_R(v\text{-}c) \neq R(v\text{-}c)$.

Lemma 3.4. Let R be a locally unital ring. If R is (von Neumann) regular then R is right (left) P-injective.

Proof. Always, $Ra \subseteq l_R r_R(a)$ for any $a \in R$. To see $l_R r_R(a) \subseteq Ra$. Let us take $x \in l_R r_R(a)$, then $r_R(a) \subseteq r_R(x)$. Write $a = ara$ for some $r \in R$ and choose v a local unit of x. Then $(v\text{-}ra)va \in r_R(a) \subseteq r_R(x)$. This shows that $x = xrava \in Ra$. The result now follows from Lemma 3.2.

Recall that a ring R is said to be *semiprime* if, for every ideal I of R, $I^2 = 0$ implies $I = 0$.

Lemma 3.5. (cf. Lemma 4.3.4 [1]). Let R be a locally unital ring which is semiprime and right P-injective. Then for every idempotent $e \in R$, the corner ring eRe is right P-injective.

Proof. Write $S = eRe$ and let $x \in l_S r_S(a)$, where x, $a \in S$. Then $r_S(a) \subseteq r_S(x)$. By Lemma 3.2, it suffices to show that $r_R(a) \subseteq r_R(x)$ (then $x \in Ra$, so $x = ex \in eRa = Sa$, as required). So let $y \in r_R(a)$, then $ay = 0 \Longrightarrow aey = 0 \Longrightarrow aeyke = 0, \forall k \in R \Longrightarrow xeyke = 0$, $\forall k \in R$ (since $r_S(a) \subseteq r_S(x)$). Thus, $xyRe = 0$ and $exy = xy$. Now consider the two-sided ideal $RxyR$ of R, and note that $(RxyR)^2 \subseteq RxyRxyR \subseteq RxyRexyR = \{0\}$. But R being semiprime $RxyR = \{0\}$ and hence $xy = 0$ (since R has local units). This completes the proof.

Recall that a ring is a right *PP ring* if every principal right ideal is projective.

It is worth mentioning that a ring R without identity may not be a projective R-module. But a Leavitt path algebra over an arbitrary graph is always projective as a module over itself (see Corollary 2.3 [3]).

We use a part of Lemma 8 [6] and note the following lemma.

Lemma 3.6. (cf. Example 5.8. [5]). Let R be a locally unital ring which is semiprime. If R is right P-injective, right PP ring then R is (von Neumann) regular.

Proof. Let a be any element in R with u a local unit of a. Since R is a right PP ring, aR is projective and so the short exact sequence $0 \to r_R(a) \to R \to aR \to 0$ splits. Writing $S = uRu$ and arguing as in Lemma 8 [6] we get that the following short exact sequence
$0 \to r_S(a) \to S \to aS \to 0$ splits. Thus, $r_S(a) = eS$, where $e^2 = e \in S$. Hence (by Lemma 3.2) $Sa = l_S r_S(a) = S(u\text{-}e)$. Now $Sa \oplus Se = S$ implies that $a \in aRa$.

Before stating our main theorem, we shall recall few more lemmas.

Lemma 3.7. (see Proposition 2.3.1 [1]). Let E be an arbitrary graph and K be any field. Then the Leavitt path algebra $L_K(E)$ is semiprime.

Lemma 3.8. (see Theorem 3.7. [7]). Let E be an arbitrary graph and K be any field. Then every one-sided ideal of $L_K(E)$ is projective.

We are now in a position to show that every right (or left) P-injective Leavitt path algebras are (von Neumann) regular.

Theorem 3.9. Let E be an arbitrary graph and K be any field. Then the following are equivalent.

(1) $L_K(E)$ is right (left) P-injective.

(2) $L_K(E)$ is von Neumann regular.

(3) E is acyclic (i.e. contains no cycle).

(4) $L_K(E)$ is locally K-matricial.

Proof. (1) $\Longrightarrow$ (2) follows from Lemma 3.7, Lemma 3.8 and Lemma 3.6. (2) $\Longrightarrow$ (1) follows from Lemma 3.4. The equivalence of (2), (3) and (4) can be seen in Theorem 3.4.1 [1].

Acknowledgment

We would like to express our gratitude to Prof. Kulumani M. Rangaswamy and Prof. Gene Abrams for fruitful conversations.

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