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An Approach to Fundamental Concepts of Mathematics I: Set, Structure, System, Model

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Abstract: In the introduction a preliminary consideration of the sense of concepts "set", "structure", "system", and "model", as well as of the connection between them is proposed and on the basis of its results the task of investigation of abstract bases of system (in particular reconsideration of concept "abstract set"), of abstract structure, abstract system and of S-Model is formulated. The first section is devoted to formulation of an approach to construction of aggregate theory regarded as an analog of moderate constructive set theory: the relations "inclusion" and "equivalence" between aggregates and the operations on aggregates "union", "intersection, "difference" and "compliment" are introduced. Definition of the concept "a-system" as well as of its a-structure on the space of aggregates is defined. A special attention is shown to similarity and essential difference between aggregate theory and set theory as well as to the fact that the famous paradoxes of Cantor and Russell in the Cantor's set theory take no place in the aggregate theory. Further a variant of Cantor's set theory called restricted discrete set theory is considered. The second section is devoted to formulation of an approach to construction of algebra of a-systems. At the end the concept of system of successive systems (SSS), is introduced.

Key words: Aggregate, set, structure, system, model.

1. Introduction: About the Connection between Concepts of Model, System, Structure and Set in the Process of Their Explication

The concepts "structure", "system" and "model" play fundamental role in the language of modern mathematics—they can be applied essentially in all branches of mathematics, moreover, they are used in various fields of scientific knowledge outside mathematics as well as in the every-day language.

As a result of this fact we can mention that these concepts obtain different sense in the various cases of their application. Likewise the corresponding terms of every-day language obtain different explications in different cases of their application.

It is necessary that this semantic diversity of the terms "structure", "system" and "model" to be overcome in two reasons:

(1) Indeed this diversity of the treatment of sense of terms "structure", "system" and "model" mentioned above is tolerable in the cases when the given reasoning involving their application is closed in the given special branch of science or if it is in a close circle of reasoning outside of science characterized with stable methods of treatments of the problems. However in the epoch of more and more domination of interdisciplinary investigations in the field of science (and eventually of the practical life), connected with joint research of complex problems by specialists of different branches of science (and eventually of practice) requires coordination of their methods of research and in the same time coordination of their languages, in particular of their terminology. So, the interdisciplinary investigation of complex problems can be achieved by means of introducing a common language with common terminology for all persons or at least by coordination of their languages (including coordination of their terminology).

(2) Taking into account the fact mentioned above, it is necessary for systematic construction of

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foundations of mathematics coordination of languages (in particular of the terminology) of its various branches.

In view of statement above, the following task is posed in the present paper: (1) to formulate very general explications of the concepts “structure”, “system” and “model” which have to serve as starting points of a process of their systematic, step by step concretization producing in such an way various explications of the corresponding terms which turn out to be essential, and (2) to formulate an approach to construction of mathematical theories of them (treated as a condition to be accepted as correct explications).

Formulating the task of present paper in a more concrete way we shall add to the formulation of the problem the following two preliminary conditions for its solution:

(1) The present semantic analysis of the concepts “structure”, “system” and “model” will be based below on the following thesis concerning their mutual connection:

- The concept “model” has relative character in the sense that it expresses a relation (which can be called “modelling relation”) between two systems: (1) the system-object X (of the model), and (2) the system-model Y (besides the term “system” is used here in a very general sense involving also systems consisting only of one element).

- Under concept “system of first degree” we understand a collection of predicates P (involving properties of elements—in the case of one-placed predicates, and relations between elements—in the case of two-placed and more-placed predicates), a collection of functions F, and a collection of operations O, besides in such a case these predicates, functions and operations will be called such ones of first degree too. If we treat predicates, functions and operations of first degree (at least some of them; together with initial elements) as elements of other predicates, functions and operations, then the last ones will be called such ones of second degree. Following a similar way of reasoning we obtain predicates, functions and operations of higher degree: 3, 4, 5 and so on. So, speaking about predicates, i.e. about properties or relations, about functions or about operations at all (i.e. without indication of a definite degree) we shall consider them to be of arbitrary degree; otherwise speaking we shall treat their degree as indefinite. Taking into account the fact that in set theory the concept of function is reduced to the concept of relation as well as the concept of operation—to the concept of function we have to make clear that it is so in consequence of the fact that the set theory is namely an extensional theory. However our treatment of these concepts here is intentional: to every operation corresponds a function and to every function corresponds a relation, besides the first element of these pairs does not turn out to be merely a kind of the second element of it. Therefore we make here essential distinction between relations, functions and operations.

- The collection of the predicates, functions and operations of a given system will be called “structure”. The maximum of the degrees of predicates, functions and operations composing the given system will be accepted as the degree of the given structure.

- The collections of the objects which are accepted to be initial elements for defining over them predicates, functions and operations of the given system are called basis of the last one. So, predicates, functions and operations of the given system are ones of first degree if and only if they are defined immediately only over elements of the basis of given system. Besides the defining a predicate, function or operation of a system to be either a part of initial element of the given system (i.e. to be a part of element of zero-degree), or to be one of first, of second and so on of higher degree has conventional character. This circumstance is in correspondence with the cognitive situation in the natural sciences—for example in the case of investigation of some problems in physics we accept of course merely conditionally the molecules as initial elements, in particular as smallest and indivisibles, however in the case of other investigations we take into consideration that the molecules consist of atoms and we accept (again conditionally) the last ones as initial elements, i.e. as smallest and indivisible. Further in the case of other physical investigations we treat the atoms as composed from elementary particles.

• And so, the system is a unity of a basis consisting of initial elements and a structure generally speaking consisting of predicates, functions and operations defined directly or indirectly over given basis, besides the boundary between elements of the basis of a given system and elements of its structure—the predicates, functions and operations in a given degree is conventional. So, this boundary has to be defined depending on conditions of the given investigation.

(2) Obviously the characteristics of models in the sense as they are treated above depend on the characteristics of the system-object, the system-model and the relation between them. Besides, the characteristics of both kinds of systems just mentioned depend on their basis and their structures. Taking into account the great diversity of the possible characteristics of bases, structures and modelling-relations it is clear that for construction of mathematical theories of structures, systems and models in the sense mentioned above, it is required to be formulate very general and very abstract concepts of the bases of systems, of their structures and of the modelling-relations.

Following the rule that the formulation of characteristics of an object presupposes formulation of characteristics of its components or its presuppositions and taking into account the mentioned above, we may formulate the following dependences:

• Defining the very general and very abstract concept of model presupposes defining a very general and very abstract concepts of system as well as to make choice of very general and very abstract modelling-relation;

• Defining a very general and very abstract concept of system presupposes defining very general and very abstract concepts of structure and of it basis.

On the basis of analysis of the task of present paper one may formulate the following topics for research and in the following order:

(1) Formulating an approach to treatment a very general and very abstract concept of basis of a system called “abstract basis of system”, abbreviated as “a-basis of system”.

(2) Using the concept of a-basis to formulate an approach to treatment of a very general and very abstract concept of structure called “abstract structure” and abbreviated as “a-structure”.

(3) Using the concept of a-basis and a-structure to formulate an approach to treatment of a very general and very abstract concept of system regarded as union of a-basis and a-structure, called “abstract system” and abbreviated as “a-system”.

(4) Using the concept of a-system to formulate an approach to treatment of a very general and very abstract concept of model treated as a relation between two a-systems, called “system-model” and abbreviated as “S-model”.

2. An Approach to Treatment of the A-Basis of Systems

2.1 Introduction

In the course of historical development of mathematics together with the increase of elements of mathematical content, on the first place of mathematical theorems, takes place also a process of augmentation of the number of types of mathematical objects as various kinds of numbers, of geometrical forms, of relations, of functions, of operations and so on which result into increase of the diversity of bases of various mathematical theories. In the same time on the basis of facts of history of mathematics we can state that the a-bases of various mathematical systems is considerably more conservative. Indeed in the course of development of mathematics from the ancient time of Thales till 19th century the initial basic elements of mathematics are of two types: points and natural numbers; during the last three decades of 19th century Georg Cantor laid the foundations of set theory and during the end of 19th century and the first half of 20th century it began to play a role of foundations (at least partially) of mathematics and so sets in abstract sense turned out able to play the role of initial elements of various mathematical systems. This circumstance obtained a visual expression by means of the fact that the systematic exposition of various mathematical theories usually begins with exposition of elements of the set theory.

Together with this circumstance however in the process of development of set theory as well as of its applications and especially in connection with discovered paradoxes during 20th century took place the tendency of reconsiderations of its foundations resulting in particular into formation of new variants of set theory different in some respects from the Cantor's set theory (cf. for examples [1], Ch. II, § 1) as well as theories in some respect analogical to the set theory — for example the theory of parts (Originally:"die Teiltheorie") proposed by Ernst Foradori (cf. [2]). One may treat as manifestation of this tendency also the formation of the so called aggregate theory introduced by the author of present paper below (cf. [3] and [4]). In principle the possibility such kind of theories to play a role of foundation of one or other mathematical system supplying it with appropriate a-basis of initial elements at present has to be treated as open. So, in order to achieve more complete embracing of various possibilities in the process of investigations of foundations of mathematics we have to state the possibilities of these theories to propose a-basis for one or other mathematical system and to serve for mathematical modelling of various natural systems and processes as well as their limits in this respect. In view of such a task we shall unify all these theories (including the standard set theory) under the general name "collection theory" and their chief objects—sets, aggregates, parts and so on will be treated as kinds of collection. However in order to achieve abbreviation and simplicity of treatment of the topic of this first section of present paper the treatment of collections will be reduced here to formulation of approaches only to aggregate and set theories.

2.2 An Approach to Construction of Aggregate Theory

2.2.1 Introducing the Concept "Atomic Aggregate"

Let us suppose that some atomic, aggregates, i.e. such kind of objects which are assumed to be indivisible into parts, and also are able to be aggregated, are given. They will be denoted as a_i, b_j, c_k, etc. (i, j, k = 1, 2, 3, …).

2.2.2 Introducing the Concept "Compound Aggregate"

We shall use the term "compound aggregate" in a sense which allows us to say: "A given compound aggregate u consists of atomic aggregates a_1, a_2, … a_n ($n \in N_1$, where N_1 means "natural order of natural numbers beginning with 1") provided $n > 1$.

2.2.3 Introducing the Relation "Inclusion of Atomic Aggregate a into Compound Aggregate u"

In such a case when the compound aggregate u consists of atomic aggregates a_1, a_2, … a_n ($n \in N_1$, where N_1 means "natural order of natural numbers beginning with 1") we shall say that there are relations of inclusion of atomic aggregates a_1, a_2, … a_n into the compound aggregate u (in such a case we shall say also that the atomic aggregates a_1, a_2, … a_n are subaggregates of the given compound aggregate u) and will denote this relation in the following way:

$$a_1 \subset u,\ a_2 \subset u,\ \dots\ a_n \subset u.$$

2.2.4 Introducing the Concept "Empty Aggregate"

We shall use the term "empty aggregate" denoted by "Ø" which allows saying: "the empty aggregate u consists of no atomic aggregate,"; otherwise speaking the notion "empty aggregate" denoted "nothing".

$$(\Box\exists(a_i)\ a_i \subset \emptyset \qquad (1)$$

2.2.5 Definition of Concept "Aggregate"

α will be called "aggregate" if and only if α is an atomic aggregate, or α is a compound aggregate, or α is an empty aggregate. So, by definition an aggregate is either compound aggregate, or atomic aggregate, or an empty aggregate.

2.2.6 Definition of Relation "Inclusion" between Aggregates

We shall say that an aggregate u is included in the aggregate v (as well as we shall say also that the aggregate u is subaggregate of the aggregate v) and will denote it as u ⊂ v, if and only if for every atomic aggregate which is included in u is valid that is included also in v:

$$(u \subset v) =_{Df} (\forall a_i)\ (a_i \subset u) \rightarrow (a_i \subset v), \quad (2)$$

besides it is accepted for empty aggregate ∅ :

$$(\forall\ u)\ \emptyset\ \subset u. \quad (3)$$

2.2.7 Definition and Axiom of Relation Equivalence between Aggregates

(1) Definition of relation equivalence between aggregates:

$$(u = v) \leftrightarrow_{Df} ((u \subset v)\ \&\ (v \subset u)) \quad (4)$$

where u and v denote aggregates.

(By definition the aggregates u and v are equivalent if and only if the aggregate u is subaggregate of the aggregate v and vice versa, otherwise speaking they are equivalent if and only if they consist of the same atomic aggregates.)

(2) Axiom of relation equivalence between aggregates. For every aggregate u it is valid:

$$u = u.$$

Corollaries:

(1) If v is compound aggregate, then: v ⊂ v.

(2) If a is an atomic aggregate, then: a ⊂ a.

(3) For the empty set ∅ ⊂ ∅ .

Proof. The corollaries (1)-(3) follow from the axiom of relation equivalence between aggregates formulated above and the definition of concept "aggregate" formulated in section 1.

2.2.8 Algebraic Operations on Aggregates

Let we accept **Ω** to consist of all atomic aggregates for which it is possible to be subject of the given conversation. It will be called "universal aggregate". Consequently for every aggregate u in the given conversation one can say that it is subaggregate of the universal aggregate **Ω** of the same conversation.

We shall define below operations on aggregates union, intersection, difference and compliment.

2.2.8.1 Definition of union of aggregates u and v denoted by "u ᴗ v"

$$((a) \subset (u \smile v)) \leftrightarrow_{Df} ((a) \subset u)\ \square\ ((a) \subset v)) \quad (5)$$

(The union of aggregates u and v by definition consists of all atomic aggregates and only of those atomic aggregates which are included in aggregate u, or in the aggregate v.)

2.2.8.2 Definition of intersection of aggregates u and v denoted by "u ∩ v"

$$((a) \subset (u \cap v)) \leftrightarrow_{Df} (((a) \subset u)\ \&\ ((a) \subset v)) \quad (6)$$

(The intersection of aggregates u and v consists of all atomic aggregates and only of those atomic aggregates which are simultaneously included in the both aggregates u and v as their subaggregates.)

2.2.8.3 Definition of difference of aggregates u and v denoted by "u ─ v"

$$((a) \subset (u - v)) \leftrightarrow_{Df} ((a) \subset u)\ \&\ \square\ (a \subset v)) \quad (7)$$

(The difference between aggregates u and v consists of all atomic aggregates and only of those atomic aggregates which are included in aggregate u and are not included in aggregate v as their subaggregates.)

2.2.8.4 Definition of compliment of aggregate u denoted by "─u"

$$-u \leftrightarrow_{Df} (\Omega - u) \quad (8)$$

(The compliment of aggregate u consists of all atomic aggregates and only of those atomic aggregates which are included in the given universum of atomic aggregates, but they are not included in the aggregate u.)

2.2.9 Introducing the Concept "Extensional Algebraic Operation" and Analogy between Aggregate Theory and Set Theory

Taking into account that operations on aggregates union, intersection, difference and compliment defined above in section 1 relate to change of their extent we shall unify them under the general name "extensional operations".

It is obvious that there is essential analogy between definitions of the various extensional operations in the aggregate theory and the definitions of respective extensional operations in the set theory—the difference is mainly in the way of expression of the definitions: in the set theory a key position places the concept "element of set" while in the aggregate theory its application is assiduously avoided and it is replaced by the use of the concept "inclusion of an atomic aggregate". By virtue of this essential analogy of definitions we may convey the properties of just mentioned extensional operations proved in set theory on the basis of their definitions to the respective operations in aggragate theory.

2.2.10 Thesis about Aggregate of Aggregates

Let us to be given aggregates u_1, u_2, ….u_n. In such a case we may construct the aggregate of these aggregates u_1, u_2, …. u_n and so we obtain the aggregate [u_1, u_2, … u_n]. Taking into account that these aggregates u_1, u_2, ….u_n are not in bags, contrary to the case with set theory, they are not separated in the more compound aggregate [u_1, u_2, … u_n] as their elements, the last one merely contains the atomic aggregates of the aggregations u_1, u_2,… $u_{n.}$ So, we obtain the equation

$$[u_1, u_2, \ldots u_n] = (((u_1 \cup u_2) \cup u_3) \ldots u_n) \qquad (9)$$

In another denotation for the general union of more than two aggregates we may write:

$$[u_1, u_2, \ldots u_n] = \mathrm{U}(u_1, u_2, \ldots u_n) \qquad (10)$$

2.2.11 Thesis about Aggregate of Subaggregates

Let we assume to denote the aggregate of all subaggregates of a given aggregate u as ASA(u). So, we obtain:

$$\mathrm{ASA}(u) = u \qquad (11)$$

or formulating the same thesis in more details:

$$\mathrm{ASA}\,[u_1, u_2, \ldots u_n] = \cup\,(u_1, u_2, \ldots u_n) \qquad (12)$$

Otherwise speaking:

$$(a_i \subset \mathrm{ASA}(u)) \leftrightarrow (a_i \subset u) \qquad (13)$$

(An atomic aggregate a_i is included in the aggregate of all subaggregates of a given aggregate u as its subaggregate if and only if it is included in the aggregate u as its subaggregate.)

2.2.12 Note about the Essential Difference between Aggregate Theory and Set Theory

The operations of formation aggregate of aggregates and aggregate of subaggregates used in sections 1 are essentially connected with the structure, therefore they will be called "structural operations". Taking into account the formulas (9)-(12) we may formulate: the structural operations in aggregate theory are essentially different from the respective structural operations in set theory—the operations of formation of a set of sets and of set of subsets of the given set.

A deep difference between set theory and aggregate theory consists in that the first is extensional in essence and use extensional method of defining its basic concepts as "Cartesian product of sets", "relation", "function" and "operation" while the second uses intentional treatment of these concepts.

2.2.13 Note on the Intentional Meaning of the Notions of Relation, Function, and Operation in the Aggregate Theory

2.2.13.1 In the aggregate theory we introduce the notion of relation taking it from the logic of predicate: by definition the relation is a two-placed or more-placed predicate, i.e. predicate concerning two or more arguments.

We shall distinguish between following two kinds of relations: (1) *abstract relations*—these are relations expressing formal characteristic of its elements connected with their form, position, quality, as for example the relation of inclusion between two aggregates—that a given aggregate u is included in the aggregate v as its subaggregate, and (2) *substantial relations*—these are the relations which are relevant to the nature of their elements and determine the interaction between its elements and the processes of their change as for example the causal-effect relation between a virus and the disease. Similar to this distinction is the distinction between abstract properties, functions and operations on the one hand and substantial properties, functions and operations on the other hand.

In this connection we can formulate the following thesis of philosophic-methodological point of view: The abstract relations determine the conditions for the action of the substantial relations. The author of present paper hopes this thesis to be of methodological significance for the mathematical modelling of natural systems and processes.

2.2.13.2 Roughly speaking, the notion of one-argument function in the space of atomic aggregates expresses the dependence of a given varible aggregate named the value (of the function) from a given varible aggregate, named argument (of the given function). Obviously to every one-argument function in the space of atomic aggregates corresponds uniquely a given binary relation between the same argument and value of the function, but they—the one-argument function and the respective binary relation have to be distinguished intentionally.

2.2.13.3 The notion of one-argument operation in the space of atomic aggregates generally speaking expresses act of change of the given aggregate resulting usually either into obtaining new atomic aggregates, or depriving of atomic aggregates, or merely change of the characteristics of the given aggregate. Obviously, to every one-argument operation it corresponds uniquely a given one-argument function (and taking into account the thesis in section 2.2.13.2 we may say also: to every one-argument operation it corresponds uniquely a given binary relation), but they (i.e. the operation, function and relation which are put in correspondence) have to be distinguished intentionally.

2.2.14 Introduction of the Notion "System" in the Space of Atomic Aggregates

We shall say that it is given a system X in the space of atomic aggregates **Ω** and will denote it as $Syst_\Omega(X)$ if and only if it is given an aggregate X in the space of atomic aggregates **Ω** (i.e. if and only if $X \subset \Omega$ is valid) supplied with aggregates of predicates P (expressing properties and relations), of functions F and of operations O,

$$Syst_\Omega(X) =_{Df} \{X_\Omega, \{P_j\}, \{F_k\}, \{O_l\}\} \qquad (14)$$

where j, k, l ϵ N_1 , N_1 denotes natural order of integers beginning with 1: 1, 2, 3, provided

$$X_\Omega \subset \Omega \qquad (15)$$

We can make distinction between two aspects of the systems: (1) composition, or the basis of the system containing the elements of which it consists, and (2) structure—the way of connections between these elements in result of which the last ones form a whole. And so: the aggregate X defined in the space of atomic aggregates **Ω** (i.e. $X \subset \Omega$ is presupposed) will be called "*basis of the system* X *in the space of atomic aggregates* **Ω**" and will be denoted as $Bs(Syst_\Omega\ X)$ if and only if it is given an aggregate of relations, functions and operations over (at the last account) the space of atomic aggregates.

So, by definition we have:

$$Bs(Syst_\Omega X) = X \qquad (16)$$

The structure of a given system consists of: (1) aggregate of predicates P_j; (2) aggregate of functions F_k; and (3) aggregate of operations O_l. Denoting the concept of structure of a given system X in the space of atomic aggregates **Ω** as $Str(Syst_\Omega X)$ we have:

$$Str(Syst_\Omega X) =_{Df} [P_j], [F_k], [O_l] \qquad (17)$$

Such a definition of the concept "system" is motivated in particular by the aspiration for generality of its formulation. In this connection we shall make distinction between systems in proper sense denoted as $Syst_{pr}$, empty system denoted as $Syst_e$ and systems in improper sense denoted as $Syst_{ipr}$ defined in the following way:

$$Syst_{pr\Omega}(X) = {}_{Df}\ (X_{\neq\emptyset}) \& ([P] \neq \emptyset\ V [F_k]_{\neq\emptyset})\ V\ [O_l]_{\neq\emptyset}) \qquad (18)$$

$$Syst_{e\Omega}(X) = {}_{Df}\ (X_{\neq\emptyset})\ \&\ ([P_j] = {}_\emptyset\ \&\ [F_k] = {}_\emptyset\ \&\ \&[O_l] = {}_\emptyset\ ) \qquad (19)$$

$$Syst_{ipr\Omega}(X) = {}_{Df}\ (X_{=\ \emptyset}\ )\ \&\ ([P_j] = {}_\emptyset\ \&\ [F_k] = {}_\emptyset\ \&\ \&[O_l] = {}_\emptyset\ ) \qquad (20)$$

It is obvious that the concepts of system in improper sense and of empty set are extensionally equivalent, but they are intentionally different:

2.2.15 Introduction of Quantitative Relations on Aggregates

We may introduce quantitative relations on aggregates in a way similar to the Cantor's way of their introduction in the set theory.

Definition 1. We say that aggregates X and Y are equal and denote it as X=Y if and only if it is possible to be established one to one correspondence between their atomic aggregates.

Definition 2. We say that the aggregate Y is greater or larger than aggregate X and denote it as Y > X if and only if the aggregate X is equivalent to a proper subaggregate of Y.

Definition 3. We say that the aggregate X is less than or smaller than Y and denote it as X < Y if and only if aggregate X is equivalent to a proper subaggregate of Y.

Thesis

$$(X < Y) \leftrightarrow (Y > X) \tag{21}$$

Proof. Eq. (21) follows immediately from Definitions 2 and 3.

2.2.16 Interpretation of Peano's System of Axioms of Natural Numbers by Means of the Language of Aggregate Theory

(1) We assume that every natural number corresponds at least one aggregate besides all aggregates corresponding to the given natural number are equal.

(2) The term "zero" in Peano's system of axioms we interpret as natural number of empty aggregate.

(3) The term "successor of natural number" we interpret as the union of an aggregate corresponding to the initial natural number and an atomic aggregate (provided that the last one was not previously a subaggregate of the aggregate corresponding to the initial natural number).

For brevity sake we shall presuppose further that a method of counting the atomic aggregates contained in an aggregate resulting with ascribing a natural number to the given aggregate interpreted as a measure of the quantity of the atomic aggregates contained in it is previously defined. So, the concept of number of a definite aggregate we use in the aggregate theory in quite usual way: the natural number of empty aggregate in accordance with the mentioned in item (1) above we denote as "0" ("zero"); its successor, i.e. the natural number of atomic aggregate, we denote as "1" ("one"); the successor of the last one, i.e. the natural number of the union ([a] ∪ [b]), provided that ([a] ≠ [b]), we denote as "2" ("two"), and so on.

2.2.17 Sum of Natural Numbers of Aggregates

Definition. Let us assume that m is the natural number of the aggregate X and n is the natural number of the aggregate Y. Then we can define the sum of the natural numbers m and n as the natural number of the union of aggregates X and Y, provided that (X ∩ Y) = ∅ .

Theorem 1.

$$(m + n) = (n + m) \tag{22}$$

Proof. Eq. (22) may be obtained from the commutative law for the union of aggregates.

Theorem 2.

$$(m + n) + k = m + (n + k) \tag{23}$$

Proof. Eq. (23) may be obtained from the associative law for the union of aggregates.

2.2.18 Distinction between Solid and Fluid Aggregates

Definition. We call *solid aggregate* such one which is characterised by constant definite composition of atomic aggregates, in the opposite case we call it *a fluid* (*processing*) aggregate. So, the fluid aggregates are characterised as aggregates with constantly changeable composition of atomic aggregates.

A special case of fluid aggregate is *the infinite aggregate*—this is aggregate which step by step is in constant process of increasing the number of its atomic aggregates (besides the concept of step is relative or conventional one). An example of such kind of infinite aggregate is the aggregate of the natural numbers. Otherwise speaking we treat the last one as a potential infinity—merely as a process on every step of which is open the possibility to add a new atomic aggregate, and in the same time we reject its treatment as actual infinity. In order to express this treatment of the infinity of the aggregate of natural numbers more distinctly we shall avoid to use the word “all” (for example: we shall avoid to say “the aggregate of all natural numbers”, which phrase hint at finished off of such aggregate) and will prefer to use instead it the word “every” (for example: “for every natural number it is valid that …”).

2.2.19 Introduction of the Notion “Power of Aggregate”

Definition 1. If m is a natural number of the (constant) aggregate X, denoting the quantity of its atomic aggregates then we shall say that m is the power of the aggregate X.

Definition 2. We shall accept that the aggregate of the natural numbers (and in the same time every aggregate which is equivalent to it) possesses a definite power, which will be denoted by ω.

Note the notion of power ω. It is important to underline that the notion of power ω expresses merely a type of processing aggregate, not a definite quantity like the meaning of notion of natural number ascribed to a constant aggregate.

2.2.20 Invalidity of Russell’s and Cantor’s Paradoxes in the Aggregate Theory

The Russell’s paradox in the set theory is connected with the question whether the set of all sets which don’t contain themselves as their elements contains itself as its proper element or not? It is merely impossible to raise such kind of question in the language of the aggregate theory because in the last one the use of term “element of aggregate” is avoided.

Cantor’s paradox is result of collision between two considerations: (1) on the one hand in accordance with the sense of the term “the set of all sets” we have to accept that the power of set of all sets is maximum, i.e. roughly speaking there is no set greater than the set of all sets; (2) on the other hand however it is proved that the power of the set of all subsets of a given set is greater than the power of the initial set and after application of this theorem to the notion of the set of all sets we obtain logically the set of all subsets of the set of all sets that possesses greater power that the set of all sets, which is contrary to the first thesis.

A similar paradox is avoidable in the aggregate theory because the power of the aggregate of all subaggregates in accordance with thesis 1.10 is merely the union of all separate aggregates playing the role of subaggregates of the aggregate of all aggregates which power is equal to the power of the innitial aggregate of all aggregates.

In view of the fact that Russell’s and Cantor’s paradoxes are avoidable in the aggregate theory we can characterise the last one as moderate constructive set theory—it is moderate because the used propositional and predicate logic is classical, not intuitionistic one.

2.2.21 Note about Terminology

It is worthy to formulate explicitly that avoiding the application of phrases about member-relations (for example of the kind: “element a is a member of the aggregate u) is not essential for the content of the aggregate theory, but it is merely a question of the choice of terminology. The essential distinction of the concept of aggregate from the concept of set consists in the following model: the given set as a whole, its elements as well as its subsets are in bag (cf. [5]) determining such a specific form of their individualisation, while the aggregate and its subaggrigates are not in any bag and consequently no such kind of their individualisation takes place.

2.3 An Approach to Reconstruction of Set Theory—Elements of Restricted Discrete Set Theory

2.3.1 Note about the Bag Model of Sets

The bag-model of sets requires not only all elements of the given set to be in a bag, but also every subset of it to be in a bag (it is clear that such bags which have simultaneously and separately to contain all subsets of a given set have to be very strange).

A special case is the bag-model of empty set—this is the empty bag, i.e. a bag without any elements in it. In accordance with the first point of the second axiom in Zermelo's axiomatic foundations of set theory (Axiom der Elementarmengen) empty set exists (cf. [6], p. 202) which in the bag-model means: empty bag exists. Besides the theorem formulated and proved after just mentioned axiom in accordance with which every set possesses at least one subset which is not element of it (cf. [6], p. 203) obtain in the bag-model the following interpretation and visual explanation: every beg representing a set contains at least one empty bag representing an empty subset, which can not be treated as an element of the given set. So, the just mentioned Zermelo's theorem can be treated as almost equivalent to the assertion that every set possesses empty subset.

2.3.2 Note about Distinction between Solid and Fluid Sets

The distinction between solid and fluid aggregates introduced in item 1.18 and some consequences from it appearing in a natural way in the aggregate theory can be transferred to set theory. So, we shall make distinction between solid sets characterised with constant definite composition of elements on the one hand and *fluid sets* characterised as inconstant variable, i.e. characterised by constantly changeable composition of elements, on the other hand. In this connection we postulate the following two restrictions:

(1) First restriction. The power, respectively the cardinal number of a set means the quantity of elements in it only provided that the given set is constant, in the case of fluid set its power, respectively its cardinal number, obtains quite other meaning—it denotes the way of processing of the fluid set.

Examples: the set of fingers of my right arm at the given moment is a solid set, so its cardinal number "five" means merely that my right arm possesses at the given moment just five fingers. However the set of all sets is a fluid set. Indeed if we form the set of all objects like stones, trees, animals and so on embracing all known objects (this set will be called "set of all sets of zero degree") in accordance with the sense of the term "set of all sets" we have to add to it the set of all its subsets because they also are treated as sets (they will be named subsets of first degree). The set obtained in such a way will be called "set of all sets of first degree". Further we have to continue the process of formation of subsets of second, third and so on degrees and by means of their adding to the "set of all sets of a given degree" to form the set of all sets of second degree, of third degree and so on to infinitum.

Taking into account this fluid character of the set of all sets it follows that it is not correct to speak about its power, respectively about its cardinal number as denoting the quantity of its elements like the cardinal numbers of solid sets (for example like the case that the cardinal number of the set of fingers of my right arm is five). In view of these considerations Cantor's paradox obtains its solution. The contradiction between statements "The set of all sets possesses maximum number of elements in respect to every other set" and "The set of all sets does not possess maximum number of elements in respect to every other set"—because the set of all subsets of the set of all sets has greater cardinal number than the set of all sets" is a result of different way of treatment of concept "set of all sets"; in the case of obtaining the first statement the concept of set of all sets is treated as denoting a solid set, namely in the sense of the concept mentioned above "the set of all sets of zero degree", but in the case of obtaining the second statement it is taken into account (at least partially) that the concept "set of all sets" is fluid—it is quite natural that different treatment of a concept results into obtaining different, even contradictory statements.

(2) Second restriction. The fluid sets with cardinal number ω do not may be treated as completed.

Corollaries:

(a) The cardinal number ω expresses potential infinity, not actual: in every moment of the process of adding one the possibility this process (of adding one) to continue is open.

(b) It is not correct to add other cardinal numbers after ω in the same sense as we may add 2 after 1 in the natural order of natural numbers.

2.3.3 Note on Introducing the Concept of A-System in the Space of Sets

The way of introduction of concept "system" in the space of atomic aggregates applied in section 1 can be transferred in to its introduction in the space of restricted discrete sets (for introduction of some general kinds of systems in the space of sets cf. 7).

2.3.4 About the Relationship between Aggregate Theory and Set Theory

The following question about relationship between aggregate theory and set theory arises: Is it a way to realize an transition from the aggregate theory to restricted discrete set theory? Here this question will be left open.

3. An Approach to Construction of Algebra of Systems

3.1 Relation of Inclusion of $Syst_\Omega(X)$ into $Syst_\Omega(Y)$

Definition. We shall say that $Syst_\Omega(X)$ is included nto $Syst_\Omega(Y)$ and will denote it as $Syst_\Omega(X) \subset Syst_\Omega(Y)$ if and only if: the following two conditions are fulfilled:

$$((X\subset\Omega) \,\&\, (Y\subset\Omega)) \rightarrow (X\subset Y) \qquad (24)$$

$$Str(Syst_\Omega X) \subset Str(Syst_\Omega Y) \qquad (25)$$

If $Syst_\Omega(X)$ is included into $Syst_\Omega(Y)$ then we shall call $Syst_\Omega(X)$ a subsystem of $Syst_\Omega(Y)$. So, saying that $Syst_\Omega(X)$ is a subsystem of $Syst_\Omega(Y)$. We presuppose that the field of $Str(Syst_\Omega X)$ i.e. the field of consisting its relations, functions and operations (provided their collections, i.e. their sets or aggregates, is not empty) is limited to X, and there is no other relations, functions and operations entirely or partially acting on X.

We can make distinction between following two kinds of relation of inclusion for systems, respectively between two kinds of subsystems:

Let we assume that $Syst_\Omega(X) \subset Syst_\Omega(Y)$ is valid. Then:

(1) If X = Y we shall say that $Syst_\Omega(X)$ is included in $Syst_\Omega(Y)$ in a weak sense and will denote it as follows:

$$Syst_\Omega(X) \subset^w Syst_\Omega(Y) \qquad (26)$$

as well as we shall say also "$Syst_\Omega(X)$ is a subsystem of $Syst_\Omega(Y)$ in an weak sens", or shortly "$Syst_\Omega(X)$ is a weak subsystem of $Syst_\Omega(Y)$".

(2) If $X \neq Y$ we shall say that $Syst_\Omega(X)$ is included in $Syst_\Omega(Y)$ in a strong sense and will denote it as follows:

$$Syst_\Omega(X) \subset^s Syst_\Omega(Y) \qquad (27)$$

as well as we shall say also "$Syst_\Omega(X)$ is a subsystem of $Syst_\Omega(Y)$ in a stong sens", or shortly "$Syst_\Omega(X)$ is a strong subsystem of $Syst_\Omega(Y)$".

At the end of present item we shall only introduce a concept more complicated than the concept of subsystem, which will be called "region"; the region X of $Syst_\Omega(Y)$ will be denoted as $RegX(Syst_\Omega Y)$. $RegX(Syst_\Omega Y)$ means the basis of RegX namely the collection (aggregate or set) X is strong subcollection of the basis of $Syst_\Omega Y$, namely of the collection Y, i.e. $X\subset Y$, $X\neq Y$, and its structure generally speaking consists of two components: (1) proper structure consisting of collection of relations, functions and operations which field is limited to the basis of the given region, i.e. to the collection X and (2) improper structure consisting of subcollection of the collection of relations, functions and operations forming the structure of $Syst_\Omega$ (Y) which are not elements of the proper structure of $Syst_\Omega$ (Y),however they are acting (entirely or partially) on the collection X.

The concept of subsystem as it is defined above may be treated as special case of the concept of region as it is just defined: the subsystem is a region the improper structure of which is empty.

3.2 Operation "Union of Systems" Denoted as $Syst_\Omega(X\cup Y)$

Definition.

$$Syst_\Omega(X\cup Y) =_{Df} Syst_\Omega(X) \cup Syst_\Omega(Y) = {}_{Df}(X\cup Y) \,\&\, (Str(Syst_\Omega X) \cup Str(Syst_\Omega Y)) \quad (28)$$

The union of several systems X_i will be denoted as $Syst_\Omega(\cup X_i)$. Now we can formulate the following:

Theorem 1. $Syst_\Omega(\cup X_i).= (\cup X_i) \,\&(\cup\, Str(Syst_\Omega\, X_i)$ (29)

Proof. The theorem follows immediately from the definition of the concept "union of two systems" formulated above and using the rule of induction.

Theorem 2. If a system takes participation in the operation union of systems then it is subsystem of the obtained unified system.

Proof. In accordance with formula (28) (expressing theorem 1 the collection (i.e. set or aggregate) playing the role of basis of the given system subjected under the action of operation union of systems is subcollection (i.e. subset or subaggregate, respectively) of the collection playing the role of basis of obtained from the action of just mentioned operation whole system—because according to formula (27) the basis of the whole system obtained in result of action of operation union of systems is a union of all collections which play the role of basis of the systems taking participation in the operation union of systems. In the same time the structure of the given system participating in the operation union of systems consists by definition of relations, functions and operations defined namely on the collection (set or aggregate, respectively) playing the role of basis of the system under question taking participation in the operation union of systems, consequently their action is limited only on this collection, respectively on this subcollection of the whole collection playing the role of bases of the obtained unified system as included of the union of structures of all systems participating in the operation union of systems, consequently again the subcollection of the whole collection playing the role of basis of unified system is closed in respect to the structure, i.e. to the collection (set or aggregate) of relations, functions and operations initially defined on the basis of the given system participating in the operation union of systems.

Speaking free one can say that the union of several systems is a mechanical, external gathering of systems into one whole besides the initial systems subjected under the action of the operation of union of systems keep themselves as subsystems of the whole system obtained after the action of just mentioned operation.

We can make distinction between the following two kinds of the operation union of systems:

(1) If X = Y we shall say that the union of $Syst_\Omega(X)$ and $Syst_\Omega(Y)$ is weak; in such a case the union of systems is reduced to union of structures of both systems.

(2) If $X \neq Y$ we shall say that the union of $Syst_\Omega(X)$ and $Syst_\Omega(Y)$ is strong.

At the end of present item we shall only mention that there is a more complicated and probably even more significant for the mathematical modelling in natural sciences operation on the systems called "synthesis of systems"—the basis of synthesised system is obtained again as union of the bases of the separate systems taking participation in the operation synthesis of systems, however the structure of synthesised system is obtainable in result of extension (at least partially) of the field of action of relations, function and operations constituting the structures of the initial separate systems participating in the operation synthesis of systems on the whole bases of the obtained synthetic system. It is obvious that initial systems participating in the operation synthesis of systems generally speaking do not keep themselves as subsystems in the obtainable synthetic system.

3.3 Operation Difference Concerning Systems

Definition. If $Syst_\Omega(X) \subset^s Syst_\Omega(Y)$, so if $X \subset Y$, $X\neq Y$, if the field of $Str(Syst_\Omega X)$ i.e. if the field of consisting its relations, functions and operations is limited to X, and there is no other relations, functions and operations entirely or partially acting on X, then the operation difference of $Syst_\Omega(X)$ from $Syst_\Omega(Y)$, denoted as $Syst_\Omega(Y - X) =_{Df} Syst_\Omega(Y) - Syst_\Omega(X)$ is allowed and its result is as it follows:

$$Syst_\Omega(Y - X) =_{Df} Syst_\Omega(Y) - Syst_\Omega(X) = ((Y - X) \& Str(Syst_\Omega Y) - Str(Syst_\Omega X) \tag{30}$$

Theorem. The operation difference concerning systems is a reverse operation in respect to union of systems.

Proof. If after union of $Syst_\Omega(X)$ and $Syst_\Omega(Y)$ immediately follows operation difference of $Syst_\Omega(X)$ from ($Syst_\Omega(X) \cup Syst_\Omega(Y)$) as well as difference of $Syst_\Omega(Y)$ from ($Syst_\Omega(X) \cup Syst_\Omega(Y)$) then we obtain the initial $Syst_\Omega(X)$ and $Syst_\Omega(Y)$ as separately given. The same result we obtain in the case of contrary order of operations union and difference, of cause provided the last one is allowed. At the end of present item we shall only mention that a more complicated operation than difference concerning systems which is significant for modelling of natural systems is the operation differentiation of systems which consists of separation of a region from the whole system and transformation of it into separate system which is connecting with the structural changes.

3.4 Operation Intersection Concerning Systems

If two regions $RegX(Syst_\Omega Z)$ and $RegY(Syst_\Omega Z)$ are given, then the operation intersection over them denoted as

$$Reg(X \cap Y)Syst_\Omega(Z) =_{Df} (RegX(Syst_\Omega Z) \cap RegY(Syst_\Omega Z))$$

is possible and the result of its action is the region of $Syst_\Omega(Z)$ which basis is $(X \cap Y)$ and its proper structure is the intersection of collections (aggregates or sets) of relations, function and operations consisting the proper structures of both regions $RegX(Syst_\Omega Z)$ and $RegY(Syst_\Omega Z)$.

So, the result of intersection of two regions of a given system can be either separation of a subsystem of the given system, or an empty system, or an improper system.

3.5 About Systems of Successive Spaces

In this last item of the section about explication of concept "a-system" I shall introduce the concept of a special kind of systems which I suppose to be very significant for mathematical modelling—this is the concept of "System of Successive Spaces", abbreviated as SSS.

Generally speaking Ω is a System of Successive Spaces if and only if it satisfies the following conditions:

$$\Omega = \cup\, \Omega^p,\ p \in N \tag{31}$$

where N_1 denotes the ordered sequence of natural numbers beginning with 1 and directed to greater numbers (so, for m, n $\in N_1$ n follows after m if and only if m<n). It follows immediately from Eq. (31) that:

$$(\forall p)\ \Omega^p \subset \Omega \tag{32}$$

$$\neg(p = q) \rightarrow ((\Omega^p \cap \Omega^q) = \emptyset \tag{33}$$

$$\text{If } p < q, \text{ then } \Omega^q \text{ follows after } \Omega^p \tag{34}$$

Defined in such a way Ω denotes merely a scheme of SSS, or speaking more precisely the whole space of a SSS, besides Ω^p, $p \in N_1$, will be called partial space.

The transition from the concept of scheme of SSS to the concept of SSS can be realised in two forms besides two types of SSS are obtainable, respectively:

- Qualitative SSS, denoted as qSSS—it is formed from a scheme of SSS by means of adding to last one a collection of connections between partial spaces, besides different kinds of connections determine different kinds of qSSS;
- Metric SSS, denoted as mSSS—it is formed from qSSS by means of adding a metric to every partial space.

The author of present paper regards SSS as unified means of modelling mathematical systems which determine their role for mathematical modelling of natural systems and processes (cf. [8]). So, examples of SSS will be given by means of their applications to treatment of various mathematical objects and systems in the next papers of this serious.

4. Conclusion Remark: About the Problem of Construction of Algebra of S-Models

The explication of concept of set (together with introducing the concept of aggregate) and of system in the two previous sections has prepared some conditions for explication of the concept “S-model”. However the formulation of explication of this concept requires also application of a system of dyadic modal logic. Therefore we postpone the treatment of the problem about explication of the concept “S-model” for the next paper of the present serious which will be entitled “An approach to fundamental concepts of mathematics II: towards of dyadic modal logic and algebra of S-models”.

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