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E. Cartan's Supersymmetry, Noncommutative Geometry and Propagation of Time in $S^7 \times R^4$ Spacetime

Sadataka Furui

Faculty of Science and Engineering, Graduate School of Science and Engineering, Teikyo University, Utsunomiya 320-8551, Japan

Abstract: Einstein claimed that one cannot define global time, and in order to formulate physical dynamics, it is useful to adopt fiber bundle structure. We define topological space E which consists of base space X and fibers $F = \Pi^{-1}(X)$, where Π is a projection of an event on the base space. Relations between initial data and final data are defined by group G and a Fiber bundle is defined as as set (*E, Π, F, G, X*).

Tangent bundle TX of real linear space X is defined by the projection π_{TX} = TX → X; (x,a) → a for any a ∈ X and a sphere S^n any non negative integer n may be thought to be a smooth submanifold of R^{n+1} and TS^n is identified as

$$\{(x,a) \in R^{n+1} \times S^n : x\cdot a = 0\}$$

Connes proposed that when one adopts non-commutative geometry, one can put two fibers at each point of X and on top of the two fibers define the initial input event and the final detection event. When one considers dynamics of leptons defined by Dirac equation, group G is given by quaternions H, and the base space X is usually taken to be S^3.

E. Cartan studied dynamics of spinors which are described by octonions or Cayley numbers which is an ordered product of two quaternions. The asymptotic form Y of this system is S^7. Cayley numbers of S^7 are defined as a 3-sphere bundle over S^4 with group S^3. Therefore in T X there are two manifolds $S^3 \times R^4$ and $S^{3\prime} \times R^4$ and the direction of propagation of time on S^3 and $S^{\prime 3}$ are not necessarily same.

We apply this formulation to experimentally observed violation of time reversal symmetry in $p\bar{p} \rightarrow t\bar{t}$ process and for understanding the result of time reversal based nonlinear elastic wave spectroscopy (TR-NEWS) in memoducers.

Keywords: Cartan's supersymmetry, non-commutative geometry, Higgs mechanism.

1. Introduction

Expression of propagation of time in physical processes contains subtle problems. Connes discussed interpretation of Lagrangians of the standard model in terms of non-commutative geometry [5-7].

Connes [6] introduced non- commutative geometry in quantum mechanics, in which on S^3 sphere of space coordinates, two fibre points of time coordinates are defined, and extended Quantum Chromo Dynamics (QCD) by adding one time coordinate which does not commutewith the ordinary time [5, 6]. In the expression of fiber bundles (*E, Π, F, G, X*) [1, 2, 13, 56], Connes defined two fibers F that are extended from a point of a base space X, and the group G that defines dynamics of leptons described by Dirac equation is expressed by quaternions ∈ *H*, and T S^3 can be identified as $S^3 \times R^4$. In the paper of Chamseddine and Connes [8] quantum physics was based on spectral triple (*A*, *H*, *D*) where *A*is thealgebra, (*H*, *D*) is the Hilbert space on which the Dirac operator operates.

But the quaternion can be a sub group of octonion O or Cayley number, and in asymptotic base space Y is S^7. An octonion has the triality symmetry, but its structure is not well understood although there are studies in string field theory of Atiyah and Witten [22] whose asymptotic form of Y is restricted to $S^3 \times S^3$, and not S^7 and as an element of *T X*, $S^3 \times R^4$ was

Corresponding author: Sadataka Furui, former professor, Faculty of Science and Engineering, Graduate School of Science and Engineering, Teikyo University, Utsunomiya 320-8551, Japan

studied. Cayley numbers of S^7 are defined as a 3-sphere bundle over S^4 with group S'^3. Therefore in $T X$ there are two manifolds $S^3 \times R^4$ and $S^{3\prime} \times R^4$. Cartan derived triality symmetry of spinors on S^3 and S'^3 that interact with 4 dimensional fields x and x'.

The definition of triality symmetry of Ref. [22] is different from that of E. Cartan, since they consider only R-symmetry and K-Symmetry among fermions, while Cartan defined fermions by spinors ξ and vectors by x and $x'$are interactions of ξ and x are considered. In Cartan's supersymmetry, there are G_{23}, G_{12}, G_{13}, G_{123} and G_{132} transformations of fermions and vectors, and except G_{23} there are transformations of ξ to x or x′and vice versa. The fermion spinors ξ has 16 bases ξ_0, ξ_i, (i = 1,···,4), ξ_{1234}, ξ_{234}, ξ_{134}, ξ_{124}, ξ_{123} and ξ_{ij},(i< j) ∈ {1,2,3,4}. Since $G_{23}\xi_4 = \xi_0$ and $G_{23}\xi_{123} = \xi_{1234}$, there are 14 independent spinor bases. Vector x and x′have respectively 4 bases (x_i (i = 1, ···, 4), $x_{i'}$ (i′ = 1,···, 4)). Since $G_{23} x_{4'} = -x_4$ and $G_{23}x_4 = -x_{4'}$ there are 7 independent bases.

Following the suggestion of Bleuler that Lorentz symmetry need not be satisfied in virtual states, we take differential of $U(1)$ defined in $S^3 \times R^4$ and $U'(1)$ defined in $S'^3 \times R^4$ opposite directions. The S matrices of such systems can be derived from induced representation and adopting semidirect product of unitary transformation and non-unitary triality transformation, that exists in Cartan's supersymmetry.

We study also superpositions of sound waves which contain nonlinear $U(1)$ component and its time reversed nonlinear wave $U'(1)$. We find interesting superposition properties of memristor wave, which can be used.

In Time Reversal violating Nonlinear Elastic Wave Spectroscopy method, presence of more degrees of freedom in superposition of time reversed focused pulses can be understood, if leptons on detectors needed to follow a quaternion which was a projected subgroup of octonions. The relation between the Quantum Electro Dynamics (QED) which was successfully described by Dirac equation and the Quantum Chromo Dynamics (QCD) which is expected to be described by Yang-Mills equation including color degrees of freedom is not so obvious. In addition to quantum mechanics, one needs to include gravitational interactions, which can be performed by changing the real degrees of freedom to the complex degrees of freedom. In quantum mechanics, two events are defined by unitary operator $U_{\alpha 1}$ and $U_{\alpha 2}$

$$U_{\alpha 1}\, U_{\alpha 2} = \sigma(\alpha_1,\alpha_2)\, U_{\alpha 1 \alpha 2}$$

andnonequivalence of $U_{\alpha 1\alpha 2}$ and $U_{\alpha 2\alpha 1}$ means violation of time reversal symmetry.

Two representations

$$U = L_1 \oplus L_2 \oplus \cdots,\ U = M_1 \oplus M_2 \oplus \cdots$$

are identical if there is bounded linear operator V that satisfies $VL_x V^{-1} = M_x$,which means

$$V L_x = M_x V$$

for $x \in H$ and x transform as $x \to hxh^{-1}$.

Induced representations [10] are useful in describing a group G which has a closed normal subgroup N. Irreducible representations of semi-direct product G is written in the formnh.

On a space S one can define a projection-valued measure P_E, which satisfies

$$U_x P_E U^{-1}{}_x = P_{[E]\,x}{}^{-1}$$

where $E \subseteq S, x \in G$.

The time is a projection valued measure, and we apply the induced representation to the study of time reversal symmetry. A superposition of a nonlinear sound wave and time reversed nonlinear sound wave can reduce improper superposition of lobes around theproper peak produced by a superposition of reflected waves [24-29, 33].

In a memory based ultrasonic transducer, which is called the memosducer, memristor based TR-NEWS (Time Reversal based Nonlinear Elastic Wave Spectroscopy), similar cancellation of nonlinear waves by time reversed nonlinear waves was observed [39-41]. There are confusions in the definition of memristance

$$M(\varphi,Q) = d\varphi/dQ$$

where$\varphi = \int U dt$, and which is identified as $d\varphi/dQ = (d\varphi/dt)/(dQ/dt) = dU/dI$[48]. It is worthwhile to reconsider behaviors of $M(\varphi, Q)$ and time dependence of charge $Q(t)$.

In particle physics, time reversal symmetry violations are detected as Charge and Parity (CP) Symmetry violation effects since Charge, Parity and Time (CPT) Symmetry is believed to be valid.

A comparison of

$$B^0 \to K^+\pi^-$$

and

$$B^0 \to K^- \pi^+,$$

$$A_{K\pi}=(n_{K-\pi+} - n_{K+\pi-}) / (n_{K-\pi+} + n_{K+\pi-})$$

measured by the BABAR collaboration is −0.088 ± 0.035 ± 0.013, which implies that the CP symmetry is violated.

Cartan's supersymmetry [3, 13, 56] that allows asymmetry of time reversal could explain $p\bar{p} \to t\bar{t}$ forward backward asymmetry, and solve the problems of decay widths of Higgs boson to vector mesons. It could also explain chaotic behaviors of memristor based ultrasonic transducer, which can be applied for realizing lossless electronic wave propagation that will be used in medical detection processes and others.

In section 2, we present the Hofstadter butterfly model to clarify essential features of non-commutative geometry and in the section 3 specific features of superposition of nonlinear time reversed wave and original nonlinear wave are shown.

In section 4 we show time reversal and non-commutative geometry can be formulated consistently using Cartan's supersymmetry.

In section 5, we try to explain experimental results of $p\bar{p} \to t\bar{t}$ forward backward asymmetry observed at Tevatron by the new framework of time reversal symmetry.

Discussion and conclusions are given in section 6.

2. Non-commutative Geometry and the Hofstadter Butterfly Model

Hofstadter [23] calculated 2 dimensional eigenvalue equation of Bloch electrons whose energy is given as

$$W(k) = 2E_0(\cos k_x a + \cos k_y a).$$

The momentum $(h/2\pi)\, k$ is replaced by $p - eA/c$

$$E_0[\psi(x + a,y) + \psi(x-a,y) + e^{-ieHax/(hc/2\pi)}\psi(x,y + a) + e^{+ieHax/(hc/2\pi)}\psi(x,y-a)] = E\,\psi(x,y)$$

Coordinates x,y are replaced by $x = ma, y = na$ and the ratio E/E_0 is replaced ϵ, and

$$\psi(ma,na) = e^{i\nu n} g(m)$$

and when $a^2H/2\pi(hc/e)$ is defined as α, the eigenvalue equation becomes

$$g(m + 1) + g(m-1) + 2\cos(2\pi m\alpha-\nu)g(m) = \epsilon\, g(m).$$

In matrix form, the above equation can be written as

$$\begin{pmatrix} g(m+1) \\ g(m) \end{pmatrix} = \begin{pmatrix} \epsilon - 2\cos(2\pi m\alpha - \nu) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} g(m) \\ g(m-1) \end{pmatrix}$$

2.1 Pattern of Time Reversal Violation and Resolution of Detectors

A requirement of the periodicity is [30]

$$2\pi\alpha(m + q)-\nu = 2\pi\alpha m-\nu + 2\pi p$$

which yields

$$\alpha=p/q$$

We define a ring A on complex C. As in fractional quantum Hall effects, elements of rings are characterized by an angle θ, and we define two generators of a ring A_θ, U and V with a rule

$$V\,U = (\exp 2\pi i\theta) UV.$$

We take $p = 1 + 2m$ and $q = 4 + 2n$ and make a table of p,q pairs which are relatively prime, and solved the eigenvalue equations exactly.

When the maximum number of q is less than about 400, p/q does not become close to 1/4 except the initial and final states, which are exactly 1/4, and combinations of pure numbers p/q are close to 1 and real part of $\exp(2\pi ip/q)$ remains larger than $1/\sqrt{2}$. In the case of 2000 combinations of prime numbers, we find a pair p/q = 127/140, which gives $\exp(2\pi ip/q)$ = 0.834573−0.550897i. When the maximum number of

q is 600, we studied 6000 combinations of prime numbers and found a pair p/q = 59/244 which is close to 1/4 and we find $exp(2\pi ip/q)$ = 0.0514788 + 0.998674i, which is nearly pure imaginary.

In a Hofstadter butterfly, eigenvalue e are plotted as a function of $\theta = q/q_{max}$, and we observe symmetry by a reflection of θ is worse in the case of p/q = 59/244 as compared to p/q = 127/140, and gaps around $\theta = 0.25$ and $\theta = 0.75$ disappears in the case of $p/q \simeq 1/4$. In Figs. 1 and 2 ordinate are θ and abscissas are e. The difference of gap suggests that in experiments of time reversal violation, time resolutions of detectors are essential factors.

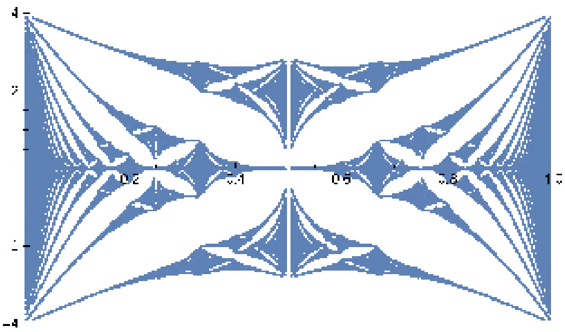

Fig. 1 Hofstadter butterfly of *q/p*=127/140.

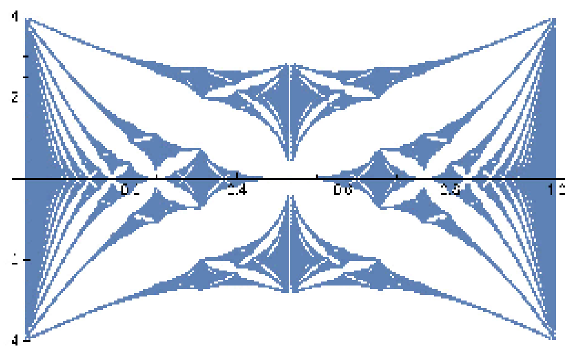

Fig.2 Hofstadter butterfly of *q/p*=59/244.

In quantum Hall effects, magnetic fields B adds phase to that of an electron [34], and θ can be irrational. Time reversal symmetry of Fractional Quantum Hall Effect(FQHE) and Chern-Simons theories is discussed in Refs. [29, 35].

2.2 Superposition of Nonlinear Time Reversed Wave and Original Nonlinear Wave

Arrays of transducers can re-create a sound and send it back to its source as if time had been reversed [25], when proper conditions on the time reversed non-linear wave are applied [24, 26, 27, 31, 32].

In superposition of time reversed electromagnetic wave in chaotic system [33], one can send linear (non-linear) port and receive linear (on-linear) port. In 1971, L. Chua cathegorized electronic elementsas resistor (R), capacitor (C), inductor (L) and memristor (M) [37, 38]. When one defines the voltage of the system by U, the resistor can be defined as $R(U,I) = dU/dI$ and by defining a flux $= \int Udt$, and inductor $L_{\varphi,Q} = d\varphi/dI$ the ideal memristor becomes

$$M(\varphi,Q) = d\varphi/dQ$$

which is identified as

$$d\varphi/dQ = (d\varphi/dt)/(dQ/dt) = dU/dI$$

It has the same dimension as the resistor but contains nonlinear effects how the charges are stored. In the Connes-Lott formulation[5, 56], one considers an even element of 3×3 matrix, i.e. $\Omega^0{}_\eta = M^+{}_3 = M_2 \times M_1$, and $\Omega^1{}_\eta = M^-{}_3$, where

$$\eta = \begin{pmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ -a_1{}^* & -a_2{}^* & 0 \end{pmatrix}$$

is gauge invariant.

In this model, the algebra is restricted to H ⊕ C where H is the quaternion, associated potential is $SU(2)_L \times U(1)$, and connection was written in the form

$$\omega = \begin{pmatrix} A_0 + A_3 & \omega_v \\ \omega_v{}^* & A_0 - A_3 \end{pmatrix}$$

whereA_0 and A_3 are $U(1)$ gauge potential. Higgs field Φ was defined as $\omega_v - \eta$, which has the doublet partner

$$\Phi' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{\Phi}$$

When associated potential is $SU(3)_L \times U(1)$, we consider O ⊕ C ⊂ $M^+{}_3$ and the same $U(1)$ gauge potential but two different Higgs fields, which allows to choose a space $S^3 \times U(1)$ and $S^3 \times U'(1)$ whose time directions are opposite.

Superposition of nonlinear waves in Memristor [37] was studied in Ref. [39], and in Ref. [41] application to crack detection using the time reversed memristor wave was proposed. When there are time reversal violations, the phase φ and the charge Q do not necessarily synchronize. There are discussions on the role of magnetic induction in memristor [48, 49]. Time reversal of acoustic or electromagnetic system can be treated as in Hofstadter butterfly. The event receiving the ordinary wave U and the time reversed wave V can be regarded as generators of A_t and UV and VU are non-commutative and additional phase factors appear. Magnitude of the phase in memosducers is expected to be small in ordinary systems.

3. Symmetry of the Space Time

As discussed in the introduction, we assume that in the asymptotic space Y of topological space of leptons is S^7 and in $T\,X$ one can define $S^3 \times R^4$ and $S^{3\prime} \times R^4$ Gupta-Bleuler [4] pointed out that Lorentz condition need not be strictly satisfied for virtual vacuum states whose time may be different from that of the observed vacuum.

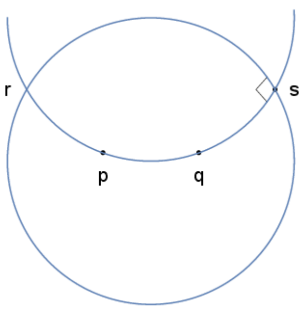

Fig. 3 The Poincaré's division of S^3 universe into an S^2 disk and an S^1 line.

The relation between the Quantum Electro Dynamics (QED) which was successfully described by Dirac equation and the Quantum Chromo Dynamics (QCD) which is expected to be described by Yang-Mills equation including color degrees of freedom is not so obvious. In addition to quantum mechanics, one needs to include gravitational interactions, which can be performed by changing the real degrees of freedom to the complex degrees of freedom.

In Poincare model of space-time, links of $S^2 \times S^1$ contained in S^3 and a point at infinity constructs S^4, which is specified by σ^i, and the metric is defined by

$$d\tau^2 = N^2(d\sigma^0)^2 - (N_i d\sigma^0 + d\sigma_i)g_{ij}(N_j d\sigma^0 + d\sigma_j)$$

and time can be defined as spacial average T = Rd3στ0 that stands on the top of a fiber.

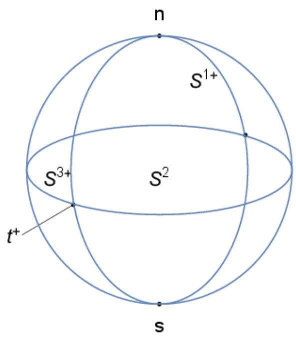

Fig. 4 Poincaré's construction of S^{3+} from S^2 disk and an S^{1+} circle.

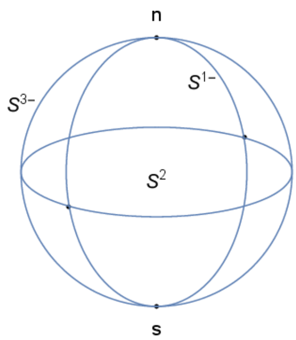

Fig.5 Poincaré's construction of S^{3-} from S^2 disk and an S^{1-}circle.

Connes and Lott[5] introduced noncommutative geometry in quantum mechanics, in which on $S3$ sphere of space coordinates, two fiber points of time coordinates are defined, and extended QCD by adding one time coordinate which does not commute with the ordinary time [6, 5]. In Connes-Lott model [5] a manifold of $O(4,1)$ symmetry, i.e. time on the added coordinate propagate in the same direction was considered. AsConnes, we assign each point on S^3, two fibers F and F′, on top of which two different times are assigned due to the non-commutative geometry. The world with two fibers at each space points on S^3 gives a new insight of time reversal symmetry. Application of the non-commutative geometry to particle and nuclear physics and quantum Hall effects was discussed in Ref. [4].

3.1 Non-commutative Geometry and E. Cartan's Supersymmetry

Using Connes' non-commutative geometry, one can take two fiber on local points of S^3, and define two points whose time propagation is defined by U(2) and Lorentz invariance is preserved, if distance ds on the manifold is defined as

$$\mathrm{ds} = \frac{1}{D}, \mathrm{ds}^2 = \gamma_{\mu\nu}\mathrm{dx}^\mu \mathrm{dx}^\nu$$

and

$$D = \frac{1}{\sqrt{-1}}\gamma^\mu \nabla_\mu$$

$$\frac{d^2x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{00}\left(\frac{\mathrm{dt}}{\mathrm{d}\tau}\right)^2$$

when

$$\Gamma^\mu{}_{00} = -\frac{1}{2}g^{\mu\nu}\frac{\partial g_{00}}{\partial x^\nu}$$

Cartan's model contains sum of $S^3 \times S^{l+}$ and $S^3 \times S^{l-}$ spinor bundles couple to vector fibers.

3.2 Theorem of Number Systems

Furwitz Theorem[9] says that for real ξ, η and ζ

$$\zeta^2{}_1 + \cdots + \zeta^2{}_n = (\xi^2{}_1 + \cdots + \xi^2{}_n)(\eta^2{}_1 + \cdots + \eta^2{}_n)$$

can be satisfied for $\zeta_1\cdots\zeta_n$ suitably chosen bilinear forms in $\xi_1,\cdots,\xi_n$ and $\eta_1,\cdots,\eta_n$, only in cases n = 1, 2, 4 and 8. The case n = 1 corresponds to real numbers R, n = 2 corresponds to complex numbers C, n = 4 corresponds to quaternion numbers H, and n = 8 corresponds to octonion numbers O. The Structure Theorem [9] says that every alternative, quadratic, real algebra without divisor of zero is isomorphic to R,C,H or O.Vector product algebra defined by $(v, u) \to (1/2)(uv-vu)$ of quaternions Im H is Lie algebra, since Jacobi identity

$$m_J(u,v,w) = u \times (v\times w) + v \times (w\times u) + w \times (u\times v) = 0$$

is satisfied, but Im O is not Lie algebra, since Jacobi identity is not satisfied in general. The Jacobi identity can be extended to Malcev identity[11].

$$m_M(u, v, w) = (u \times v)\times(v \times w) + u \times [(u \times v) \times w] - u \times [u \times (v \times w)] + v \times [u \times (u \times w)] = 0$$

if $u \times v = -v \times u$, and $u, v, w \in$ O and also $u, v, w \in$ R, C, H. The product $(v, u) \to (1/2)(uv-vu)$ are the only vector product algebra and there is no infinite dimensional vector product algebra. There is no contradiction with non-commutative algebra. We adopt the quaternion structure over complex vector space as Crumeyrolle [12], whose spinor structure with triality has the torogonal spinor structure. The structure is different from that of Connes, De Nittis and Gomi [14, 15] whose quaternion structures are fixed by involutions.

4. Symmetry of S Matrices

Cartan[3] considered a pair of Dirac spinors with transformations of the 4th component of some spinors and transformations of a pair of 4 dimensional vectors via the triality transformation G_{23}. The triality transformation G_{12} and G_{13} on some time components of the spinor induce vectors and defined

$$\psi^T C\psi = -\xi_1\xi_{234} - \xi_2\xi_{314} - \xi_3\xi_{124} + \xi_4\xi_{123}\,\xi_4$$

and ξ_{123} corresponds to the large and small component of the time component of a spinor U(1), and

$$\varphi^T C\varphi = \xi_0\xi_{1234} - \xi_{23}\xi_{14} - \xi_{31}\xi_{24} - \xi_{12}\xi_{34}.$$

ξ_0 and ξ_{1234} corresponds to the upper and lower component of the time component of a spinor $U'(1)$ which are the pair of $U(1)$ in non commutative geometry. He defined real vectors $X_0, X_1, \cdots, X_7$ as a linear combinations of x_i (i = 1, ...,4) and their complex conjugates x_i*, and a scalar F, as follows:.

$$x_1 = X_0 + iX_1, \quad x_2 = X_4 + iX_3,$$

$$x_3 = X_6 + iX_2, \quad x_4 = X_7 + iX_5$$

$$x_1' = X_0 - iX_1, \quad x_2' = X_4 - iX_3,$$

$$x_3' = X_6 - iX_2, \quad x4' = X_7 - iX_5$$

$$F = x_1x_1' + x_2x_2' + x_3x_3' + x_4x_4'$$

$$= X_{02} + X_{12} + X_{22} + X_{32} + X_{42} + X_{52} + X_{62} + X_{72}$$

Couplings of vectors x_i (i = 1,2,3), to ψ are given by $x_i\xi_i\xi_{1234}$ and those to φ are given by $x_i\,\xi_{i4}\xi_{123}$. The spinors ξ_{1234} and ξ_{123} do not rotate ξ_i or ξ_{i4} and can be considered as center elements of octonions.

Cartan's supersymmetry fixes couplings of two types of fermions ψ, φ and two types of vector fields E and E'. We assume that the coupling of Higgs bosons to two leptons and to two quarks are given by Ref. [16].

$$-y^{ij}_{\ell}E_i(L_j \circ H_d) = -y^{ij}_{\ell} H^0_d\bar{\ell}L_i\ell_j$$

and

$$-y^{ij}_{b}D_i(Q_j \circ H_d) = -y^{ij}_{b}v_d\bar{b}_ib_j.$$

where the first term yields $Z\ \ell\bar{\ell}$ coupling and the second term yields $W\bar{\nu}$coupling. We assume lepton number and quark number are conserved. Dirac spinors consist of 2 × 2 components, and each component transforms by quaternions.Cartan's spinors consist of 4 × 2 components, and they transform not by quaternions but by a kind of octonion.

Since Jacobi identity is not satisfied for Im O $\ni i,j,k,p$

$$p\times(i\times j) + i\times(j\times p) + j\times(p\times i) = -3(p\times k)\ 6 = 0,$$

octonions do not form Lie algebra but any base of octonionse_1 whose square is -1 generates a subalgebra of O isomorphic to C, and one can choose e_2 that anticommutes with e_1 and e_1, e_2 generates a subalgebra H, which forms Lie algebra. With an octonione_4 whose square is -1 and anticommutes with e_1, e_2, e_1e_2 one can construct octonions, and the e_1, e_2, e_4 is called a basic triple.

Supersymmetry in five dimensional fermion fields and four dimensional vector fields have the same structure as that of Atiyah and Witten [21, 22]. InRef. [22], the manifold of

$$X_{I,\Gamma} = S^3 \times \mathrm{R}^4/\Gamma,$$

where Γ is a finite subgroup of SU(2), they introduced G_2holonomy. Cartan's manifold contains Atiyah-Witten's manifold or G_2 manifold which is 8 + 3 + 3 dimensional and contains symmetry of octonions.

Since Cartan's octonions are not Cayley numbers, but have triality symmetries, Cartan's supersymmetry was consistent with tree level Higgs boson dynamics [42-46]. There are particle models based on non-commutative geometry which are consistent with standard model [5]. In their model, transformations by quaternion bases are adopted, but we can extend transformation by octonions and incorporate triality transformations which are $C\ell_8 \to C\ell_8$ transformation of order 2. For example, quarks $\xi_{14}\xi_{123}$ couple to x_1, and $\xi_{34}\xi_{123}$ couples to x_3 and the two coupling events are treated as non-commutative events, and couplings of ξ_{14} to ξ_{34} via a component of Higgs scalar $\xi_{ijk}*\xi_{ijk}$ and complex vector field $x_1+i\ x_3$become possible. Quarks $\xi_1\xi_{1234}$ couple to x_1 and $\xi_2\xi_{1234}$ couple to x2 and transition of ξ_1 to ξ_2 via Higgs scalar $\xi_i*\xi_i$ and complex vector field $x_1+i\,x_2$ becomes also possible.

Cartan's supersymmetry is not based on non-commutative geometry and does not contain each pair particle of involution. It is based on 2 three + one dimensional leptonic spinors and 2 four dimensional vector spaces, which transform with each other. We expect that vector fields generated by G_{12} (A,B) cannot be detected by detectors whose fermions are produced by G_{12} (E,E') [43, 46]. Cartan'soctonion consists of basic triples, which can be expressed as$S^3\times$C. Cartan defines a scalar field f which transforms as

$$f \to g^{-1}fg$$

and appropriate derivative

$$D_if = e_if + [\omega_i,f] = (e_i + \omega_i^a e_a)f.$$

Let $C(V)$ be the algebra of smooth functions on spacetime and M_n a matrix algebra that expresses possible internal structures and $A = C(V) \times M_n$ and introduce derivations

$$\tilde{e}_\alpha = e_\alpha + A^a{}_\alpha e_a$$
$$\tilde{e}_r = e_r + \omega_r{}^a e_a = \Phi^a{}_r e_a$$

where we define $\omega_r{}^a + \delta_r{}^a = \Phi^a{}_r$ as a Higgs field, and the gauge transformation $\tilde{e}_I \to g*\tilde{e}_i = g^{-1}\ \tilde{e}_ig.$

The gauge transformation in $S^3 \times R^4$spacetimeω and ω' consists of horizontal part A and vertical part $\omega_v$which contains the freedom of the choice of a fiber corresponding ω_v and ω'_v, where we define$\omega'_r{}^a + \delta_r{}^a = \Phi'_r{}^a$ and express

$$\omega = (A^a{}_\alpha\theta^\alpha + \omega_r{}^a\theta^r)\lambda_a$$
$$\omega' = (A^a{}_\alpha\theta^\alpha + \omega'_r{}^a\theta^r)\lambda_a$$

Here θ is defined from an element f of non-commutative algebra A as

$$e_i f = [\lambda_i, f].$$

and e_i belongs to a family of Lie algebra

$$df(e_i) = e_i f$$

one can construct a frame $\theta^i(e_j) = \delta_j^i$ and $\theta = -\lambda_i\theta^i$ and write exterior derivative df of an element of A as

$$df = -[\theta, f]$$

From fiber points of Φ and Φ' which corresponds to the Higgs degrees of freedom at one time, one can define orbits which are defined by horizontal part of ω and when A is described by octonions and enough degrees of freedom, dynamics in the world of non-commutative geometry projected on the $S^3 \times U(1)$ bundle remains same as that of commutative geometry. Octonions in R^7 can be defined as $R \oplus R^{0,7}$ where $e^0 \in R$, and in $R^{0,7}$, we define orthonormal vectors e_i, (i = 1,···,7) and trivector v as

$$v = e_{124} + e_{235} + e_{346} + e_{457} + e_{561} + e_{672} + e_{713}$$

and the product of octonions $a,b \in O$ is expressed in terms of the Clifford product as [12]

$$a \circ b = \langle ab(1-v)\rangle_{0,1}.$$

The four vector $w = ve^{-1}_{12\cdots7}$ is defined as

$$w = e_{1236} - e_{1257} - e_{1345} + e_{1467} + e_{2347} + e_{2456} - e_{3467}$$

Octonions satisfy the G_2 symmetry whose transformation is defined by 1 + w and 7 − w and related to algebra on S^6[13].

5. Time Reversal Symmetry Violating Processes

Nambu-Goldstone theorem of gauge theories says that the vector bosons W and Z are produced by conversion of mass-less bosons. Experimentally, Higgs boson decay branching ratio to $W\overline{W}$ shows relatively large differences from the standard model, and $p\bar{p} \to t\bar{t}$ in which couplings of W to $\bar{t}-b$ and $\overline{W}$ to $t-\bar{b}$ play roles, there appears time reversal violating effects. The quantum electrodynamics, in which fermions like e,μ,τ are described by Dirac spinors and bosons like photons are described by vector fields was quite successful, but structure of u, d, s, c and b, t quarks are not well known. Brodsky et al pointed out that the renormalization following Principle of Maximal Conformality reduces the discrepancy of magnitudes of $p\bar{p} \to t\bar{t}$ reactions between standard model results and experimental results [17, 20]. In t-quark Higgs coupling propagation of Cartan's spinor ξ_{123} contributes, while in $\bar{t}$-quark Higgs couping propagation of Cartan's spinor ξ_{1234} contributes in super symmetric systems. If propagation of ξ_{1234} is suppressed as compared to the propagation of ξ_{123} forward backward asymmetry of $p\bar{p} \to t\bar{t}$ can be explained.Cartan's supersymmetry gives relation between oscillations of fermion ($\xi_0, \xi_4, \xi_1, \xi_2, \xi_3$), and antifermion ($\xi_{123}, \xi_{1234}, \xi_{14}, \xi_{24}, \xi_{34}$) of 2+3 dimensions and 4 dimensional vector fields (x_1, x_2, x_3, x_4) and (x'_1, x'_2, x'_3, x'_4).

It is possible to make our theory consistent with the standard model by ignoring the differences of center spinor elements of particles $\psi^i\xi_{1234}$ and those of antiparticles ξ_{123}, since

$$[\xi_{1234}, \xi_i] = 0, (i = 1, 2, 3, 4)$$

$$[\xi_{123}, \xi_{i4}] = 0, (i = 1, 2, 3)$$

are satisfied, and adjusting Higgs boson coupling parameters y^{ij}_ℓ and y^{ij}_b appropriately. Gauge dependence of y^{ij}_ℓ and y^{ij}_b can be understood as ghost degrees of freedom [56]. The interactions including ξ_{1234} and ξ_{123} are gauge dependent, and they can be regarded as ghost [56]. The large forward- backward asymmetry $A_{FB}(M_{t\bar{t}})$ can occur through suppression of forward production of $\bar{t}$ can be due to difference of the strength of b quark ξ_{1234} and $\bar{b}$ quark ξ_{123}

5.1 b W → t and b W → $\bar{t}$ Processesthrough Higgs Propagation

In this section we calculate simplest diagrams of t quark decays to $\bar{b}$ and W using Higgs boson propagation between t and ℓ in W boson, using Cartan's supersymmetry [3].

A minimal set of top anomalous coupling to bWetc are given in Ref. [17], but there are place for new physics including Higgs boson. A W boson decay into

$\ell\bar{\nu}$, $\ell\bar{\nu}$ or $q\bar{\nu}$, $q\bar{\nu}$, and since leptonic number currents and baryonic number currents are conserved, we consider a b quark changes to a ν by weak interaction, and absorbed by W or a $\bar{b}$ quark changes to a $\bar{\nu}$by weak interaction and absorbed by W. Experimental decay width of $W \to \ell\nu$ is about 11%, where ℓ denotes left handed e,μ and τ and ν denotes right handed neutrinos, and $W \to$ hadrons which contains $W \to \bar{q}\nu$ and $W \to q\bar{\nu}$ is about 67% [19]. We consider the decay process in the lowest order without using Feynman diagrams. Top (bottom) quarks with quaternion momentum Q_i are written as $t(b)(\xi_i, Q_i)$. Anti-top (bottom) quarks with quaternion momentum Q_j are written as $\bar{t}$ ($\bar{b}$)(ξ_{j4},Q_j). Neutrinos and anti-neutrinos which appears as a component of W boson are written as $\nu(\xi_i,Q_i)$and$\bar{\nu}(\xi_{j4},Q_j)$. W can be replaced by other particles observed in jets. Brodsky et al. [18] studied $q\bar{q} \to \gamma \to t\bar{t}$and $q\bar{q} \to Z^0 \to t\bar{t}$, but not succeeded in explaining A_{FB} in the region of $M_{t\bar{t}}$ > 600 GeV.

5.2 $b\bar{b}$ $W\bar{W}$ $\to t\bar{t}$ and $\ell\bar{\ell}$ $Z\bar{Z}$ $\to t\bar{t}$ Processes

Collider independent t $\bar{t}$ forward backward asymmetries for $u\bar{u}$, $d\bar{d} \to t\bar{t}$ productions are studied in Ref. [53]. Although b and $\bar{b}$are are not in partons, they could be created in high energy collider experiments. Using the information of amplitudes of of $t \to b\bar{W}$ and $\bar{t} \to bW$, it is possible to study amplitudes of $b\bar{b}\bar{W}$ $W \to t\bar{t}$, by combining the loop of $b\bar{t}W$ and $t\bar{b}W$ amplitudes connected by Higgs bosons as shown in Fig.6 and 7, as well as $\ell\bar{\ell}$ $Z\bar{Z}$ $\to t\bar{t}$ and ℓ $t\bar{Z}$and $\ell\bar{t}$ Z or $q\bar{q}$ Z, amplitudes connected by the Higgs boson as shown in Fig.8.

Experimental data of $A_{FB}(\Delta y)$ indicates that in the region of $\Delta y = y_t - y_{\bar{t}} < 0$, the production is suppressed and in the region of $\Delta y = y_t - y_{\bar{t}} > 0$, the production is enhanced. It means that $\bar{t}$ production through ξ_{1234} propagation is suppressed, and t production through ξ_{123} propagation is enhanced.

Recent CERN CMS collaboration's experimental result of $t\bar{t}$ production and decay in pp collision at $\sqrt{s}$ = 8 TeV [20], shows that in lepton + jet channels CP violation was not observed and consistent with the standard model, but CP violation in quark + jet channels is not measured.

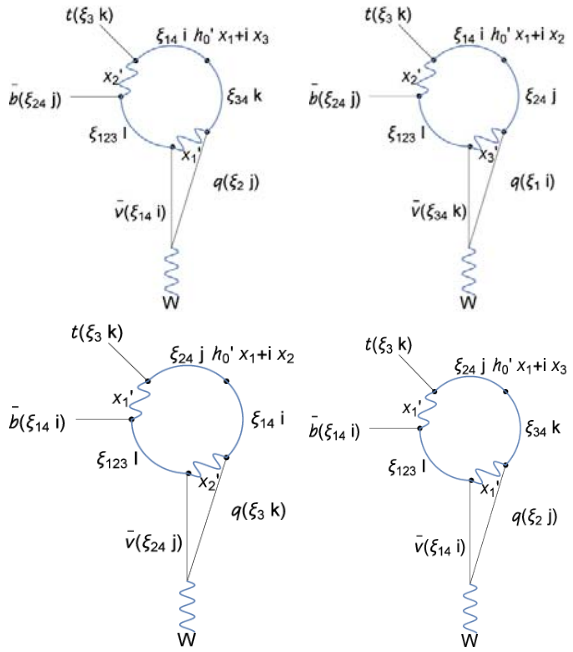

Fig.6 $W(q,\bar{\nu})$ + $\bar{b}$ $\to$ $t(\xi_3)$. Diagrams of production of $t(\xi_1)$ *and* $t(\xi_2)$ are similar.

The Nambu-Goldstone theorem of gauge theories says that the vector bosons W and Z are produced by conversion of mass-less bosons. Coleman and Weinberg [54] showed that by the Higgs mechanism, vector mesons W and Z have a massive Goldstone partner φ, defined by gauge fields v and η, or the Higgs boson in standard model is parametrized as [51]

$$\Phi = \frac{1}{\sqrt{2}}\left(1 - \frac{t}{v}\tau^i\chi^i\right)\begin{pmatrix}0\\ v+\eta\end{pmatrix} + \cdots$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}-i\chi_1/v - & \chi_2/v\\ 1 & i\chi_3/v\end{pmatrix}(v+\eta)+\ldots$$

Experimentally conservation of the leptonic current

$$J^\mu{}_\ell = J_1{}^\mu + J_2{}^\mu$$

and the baryonic current

$$J_b{}^\mu=(1/3)(J_3{}^\mu + J_4{}^\mu + J_5{}^\mu)$$

are well satisfied, but conservation of hypercharge currents

$$J^\mu{}_{\ell 5} = J_1{}^\mu - J_2{}^\mu = \bar{\nu}\gamma^\mu\gamma_5\nu + \bar{\ell}\gamma_\mu\gamma_5\ell$$

and

$$J^{\mu}{}_{b5} = J_3{}^{\mu} + J_4{}^{\mu} - J_5{}^{\mu} = \bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d$$

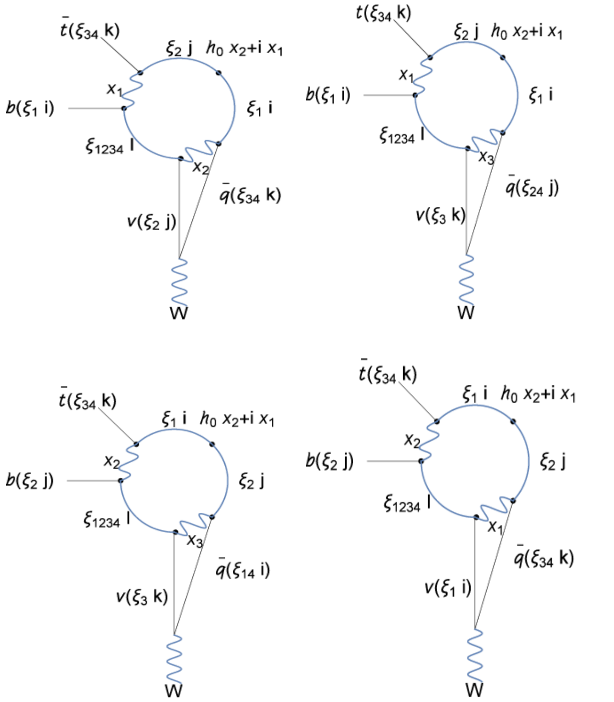

Fig. 7 ***W*(*q*, $\bar{\nu}$) + *b*→$\bar{t}$(ξ₃₄). Diagrams of production of *t*(ξ₁₄) and $\bar{t}$(ξ₂₄) are similar.**

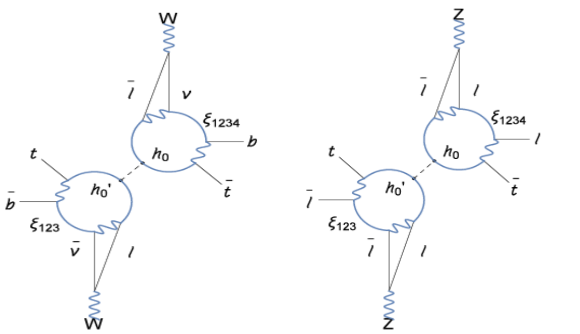

Fig. 8 b$\bar{b}$W(q,$\bar{\nu}$)$\overline{W}$($\bar{q}$,ν) → t$\bar{t}$, b$\bar{b}$W(q,$\bar{\nu}$)$\overline{W}$($\bar{q}$,ν) → t$\bar{t}$ (left) and ℓ$\bar{\ell}$Z(ℓ, $\bar{\ell}$) $\overline{Z}$($\bar{\ell}$,ℓ)→t$\bar{t}$ (right)

are not verified [55]. We expect differences are due to phases that originate from non-commutative geometry, or presence $U(1)$ and $U'(1)$ transformations.

6. Discussion and Conclusion

We showed that $S^3 \times R^4$ and $S^{3\prime} \times R^4$ that appear in the treatment of fermion in $T\,X$ represented by octonions but not quaternions and adopting non-commutative geometry which allows creation of a phase θ by exchange of generators U and V of a ring A_θ allows a uniform understanding of physics related to time reversal symmetry. Coupling to vector field X is described by $\varphi^T XC\psi$. In particle physics, Cartan's supersymmetry contains triality symmetry and more general than Atiyah Witten's supersymmetry. Scalar component of Cartan contains $\xi_0\xi_{1234} \in \Phi$ and $\xi_4\xi_{123} \in \Psi$, and couplings to vector fields X and $X'$are expressed by $\varphi^T CX\psi + \varphi^T CX'\psi$, which contains $x^1(-\xi_{14}\xi_{123} + \xi_{1234}\xi_1)$ as an example [3].

In studies of Chua's electronic circuits, presence of two fibers on $S^3 \times R^4$ and $S^{3\prime} \times R^4$ and an orbit on one space touches an orbit on another space at one point allows suppressions of noise. Using the memristor ability to enhance nonlinear signals and reduce noise by superposition of waves in a certain time domains, Time Reversal (TR) based Nonlinear Elastic Wave Spectroscopy (NEWS) methods is adopted to measure local complex damaged systems[41]. Differences of propagation of a spinor ξ_{123} and ξ_{1234} are expected to occur also in chaotic systems.

Forward backward asymmetry of $t\bar{t}$ production in $p\bar{p}$ and pp collision [20] is expected to be due to violation of CP or T symmetry, and detailed analysis are performed in Ref. [18].

To conclude, in order to evade the no go theorem [50], importance of supersymmetry was realized [52]. But it is not evident that super symmetry of Connes based on non-commutative geometry, and unique time direction of differential manifold can be chosen, and the same problem remains in the Tomita-Takesaki-Connes' theorem [57]. By using the induced representation of Mackey [10], the triality symmetry of Cartan's supersymmetry which contains time propagation in opposite direction can be incorporated both on non-communtative geometry bundles and on commutative geometry bundles.

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