Analytical Modeling of Fluid Flow through a Deformable Tube

Abdessamad Mehdari[1], Mohammed Hasnaoui[2] and Mohammed Agouzoul[1]

1. Mechanical Engineering Department, Mohammadia School of Engineering, Mohammed V University, Rabat 10 090, Morocco

2. Structure and Material Department, Royal Air Force School, Marrakech 40 000, Morocco

Abstract: In this document, we analytically investigate fluid flow through a deformable tube. The fluid is considered to be Newtonian, incompressible and it moves along an elastic and isotropic tube wall. The study provides a review of recent modeling aimed at understanding the effects of fluid parameters over the elastic wall behavior. The unsteady fluid flow will be analyzed following the asymptotic approach process using to a large Reynolds number and a small aspect radio. Moreover, according to the small axisymmetric deformation, the wall is mathematically developed basing on the thin shell theory whose linear approach is applied. Lastly, the dynamic behavior of the tube is represented and discussed.

Key words: Analytically, study, fluid, deformable, tube.

1. Introduction

Studying a fluid flow through a deformable tube is due to its occurrence for its diversity practice in many industrial systems and its capability to generate a variety of instabilities as using a rigid wall has been reported by Mehdari et al. [1]. This standing is reflected in biology by Grotberg et al. [2], in micro-fluidic devices by Squires et al. [3] and Eggert et al. [4] and in the renewable energies by Babarit et al. [5]. Recently in the field of transporting gaseous materials under pressure, Matin [6] studied fracture resistance assessment of pipes and in engineering Gilmanova et al. [7] and Greenshields et al. [8] have simulated fluid structure interaction of flexible thin shells.

Although much numerical and experimental progress has been made during the past decades, studying interaction between structure and fluid analytically is not absolutely understood yet and remains to be discovered.

The present work focuses an analytical modeling offluid flow through a deformable tube. It consists of analyzing the fluid flow aspect and its effect on the

wall tube behavior. It is based on asymptotic approach followed by a numerical simulation. The small parameter ε characterizing the aspect ratio of the tube governs the fluid asymptotic expansion. Moreover, based on linear approach of the thin shell theory, the treatment of the wall tube equations motion is developed by asymptotic process founded on geodesic curvature parameter β_2.

The rest of this paper is organized as follow. In Section 2, a formulation of the problem is presented. The linearization process is applied to make an analytical solution in Section 3. An application with interpretation is given in Section 4 and finally, conclusions are drawn in Section 5.

2. Formulation of the Problem

2.1 Governing Equations of Fluid Dynamics

In the presence of gravity force, we analyze an unsteady flow of an incompressible, viscous and Newtonian fluid. ρ and ν denote respectively the fluid density and the kinematic viscosity, L is the tube length, h is the thickness and R_0 is the radius at rest. $R(z',t')$ is the variable radius (radius is a function of the longitudinal variable and time).

Corresponding author: Abdessamad Mehdari, Ph.D., engineer (aircraft maintenance), research field: fluid flow through an elastic pipe.
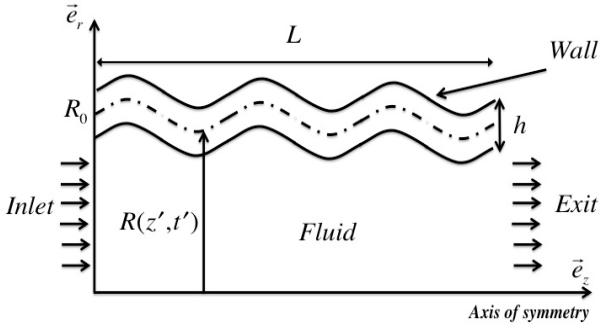
Fig. 1   The deformed domain.

We assume that the tube behaves as a homogenous and linear elastic shell with ρ_T being the tube density (Fig. 1).

The physical variables are denoted using primes. At this level, we introduce dimensionless variables, namely:

$$\begin{cases} r = \dfrac{r'}{R_o} \quad , \quad z = \dfrac{z'}{L} \quad , \quad t = \dfrac{t'}{T_f}, \\[2mm] u = \dfrac{u'}{\varepsilon W_o} \quad , \quad w = \dfrac{w'}{W_o} \quad , \quad p = \dfrac{p'}{\rho \varepsilon^2 W_o^2} \end{cases} \quad (1)$$

here, r, z and t are respectively radial displacement, axial displacement and time. u, w and p denote radial velocity, axial velocity and pressure.

The above dimensionless parameter accompanied by prime describes the corresponding physical parameter (with dimension).

Also, T_f is the reference time, $\varepsilon = R_0 / L$ is the aspect ratio and W_o represents the inlet axial velocity.

Using system (1), the dimensionless Navier-Stokes and continuity equations of the problem, are read as:

$$\begin{cases} (\dfrac{S_t}{\varepsilon}) \dfrac{\partial u}{\partial t} + (u \dfrac{\partial u}{\partial r} + w \dfrac{\partial u}{\partial z}) = -\dfrac{\partial p}{\partial r} \\[2mm] + (\dfrac{R_e^{-1}}{\varepsilon}) \left\{ \dfrac{1}{r} \dfrac{\partial}{\partial r}(r \dfrac{\partial u}{\partial r}) + \varepsilon^2 \dfrac{\partial^2 u}{\partial z^2} - \dfrac{u}{r^2} \right\} \\[2mm] (\dfrac{S_t}{\varepsilon}) \dfrac{\partial w}{\partial t} + (u \dfrac{\partial w}{\partial r} + w \dfrac{\partial w}{\partial z}) = -\varepsilon^2 \dfrac{\partial p}{\partial z} \\[2mm] + (\dfrac{R_e^{-1}}{\varepsilon}) \left\{ \dfrac{1}{r} \dfrac{\partial}{\partial r}(r \dfrac{\partial w}{\partial r}) + \varepsilon^2 \dfrac{\partial^2 w}{\partial z^2} \right\} \\[2mm] \dfrac{\partial u}{\partial r} + \dfrac{\partial w}{\partial z} + \dfrac{u}{r} = 0 \end{cases} \quad (2)$$

At large Reynolds number $\Re_e = \dfrac{R_0 W_0}{\upsilon}$ and low Strouhal number $S_t = \dfrac{R_0}{W_0 T_f}$, the system (2) is valid under the asymptotic restrictions:

$$\begin{cases} \dfrac{S_t}{\varepsilon} \equiv O(1) & \quad \dfrac{R_e^{-1}}{\varepsilon} \equiv O(1) \end{cases} \quad (3)$$

These interactions are the key to respectively analyze the coupling between the timescales and the fluid nature with space scales.

2.2 Equation of Motion of the Tube Wall

According to the thin shell theory used by Anliker et al. [9] to develop analytically shell dynamic equilibrium filled by fluid, the deformable tube is modeled by using the non-linear Kirchhoff-Love geometrical assumptions that have been reported by Frey et al. [10] and Callegari et al. [11].

The following system formulates the dimensionless variables and parameters.

$$\begin{cases} \varepsilon_0 = \dfrac{h}{R_0} \quad \bar{R} = \dfrac{R}{R_0} \quad t = \dfrac{t'}{T_T} \\[3mm] \varepsilon_1 = \dfrac{h}{L} \quad \gamma_f^0 = \dfrac{\rho . W_0^2}{\lambda_1 + \lambda_2} \quad \beta_2 = \dfrac{\bar{u}_{0.2}}{L} \\[3mm] \bar{P}_3^* = \dfrac{P_3^*}{\rho . W_0^2 \varepsilon^2} = \dfrac{P_{int-tube} - P_{ext-tube}\big|_{r'=R(z',t')-\frac{h}{2}}}{\rho . W_0^2 \varepsilon^2} \\[3mm] \gamma_c = \dfrac{\rho_T . L^2}{(\lambda_1 + \lambda_2) . T_T^2} \quad H = \dfrac{\lambda_2}{\lambda_1 + \lambda_2} \end{cases} \quad (4)$$

where, $\bar{P}_3^*$, P_3^*, $P_{ins-tube}$ and $P_{out-tube}$ are respectively dimensionless pressure gradient, pressure gradient, pressure inside the tube and pressure outside the tube. T_T is the time reference $(T_T = T_f)$ and $\bar{R}$ is dimensionless variable radius. λ_1 and λ_2 are the Lame constants and $\bar{u}_{0.2}$ is the wall axial displacement reference.

ε_0, ε_1, γ_c and γ_f^0 are constants characterizing the problem.

In this section, we adopt the "Least Degeneration Principle" approach at better modeling the tube wall behavior. The Least Degeneration Principle states as following asymptotic constraints.

$$\left\{ \quad \varepsilon^2 \equiv \varepsilon_0 \quad \beta_2 \equiv \frac{1}{R}\frac{\partial \overline{R}}{\partial z} \quad \varepsilon_1^2 \equiv \beta_2 \right. \quad (5)$$

where, $\frac{1}{R}\frac{\partial \overline{R}}{\partial z}$ is the tube geodesic curvature along the fluid flow direction.

According the system (4), the governing equations for the tube motion are stated as the under system:

$$\left\{ \begin{array}{l} \beta_2 H \dfrac{\partial \overline{u}_2^*}{\partial z} - \beta_2^2 \overline{u}_2^* + \gamma_c \beta_2^{1/3} \overline{R} \dfrac{\partial^2 \overline{R}}{\partial t^2} \\ -\left(1 - \dfrac{1}{R}\right) - \gamma_f^0 \overline{R} \overline{P}_3^* = 0 \end{array} \right. \quad (6)$$

$\overline{u}_2^*$ is the dimensionless wall axial displacement.

3. Linearized Problems and Resolutions

This section illustrates the process of solving the equations for fluid motion. In fact, we are considering linear functions for equations in system (2). These functions will be described around the particular solution at the inlet of tube. Denoting by ε the linearization parameter, the Least Degeneration Principle provides us the following form:

$$u = \varepsilon^3 \overline{u} \; ; \; w = 1 + \varepsilon^3 \overline{w} \quad p = \tilde{P}_{amb} + \varepsilon^3 \overline{p} \quad (7)$$

where, $\overline{u}, \overline{w}, \overline{p}$ are, respectively, the perturbed radial and axial velocities, and pressure. $\tilde{P}_{amb}$ is fluid ambient pressure.

$$\overline{u} = \overline{w} = \overline{p} = 0 \quad for \quad t = r = z = 0 \quad (8)$$

Inserting system (4) into system (2), we obtain at order ε^2 included, the non-degenerate equations, namely:

The 0th order terms

$$\left\{ \begin{array}{l} \dfrac{\partial \overline{u}}{\partial t} + \dfrac{\partial \overline{u}}{\partial z} = -\dfrac{\partial \overline{p}}{\partial r} + \left\{ \dfrac{1}{r}\dfrac{\partial}{\partial r}(r\dfrac{\partial \overline{u}}{\partial r}) - \dfrac{\overline{u}}{r^2} \right\} \\ \dfrac{\partial \overline{w}}{\partial t} + \dfrac{\partial \overline{w}}{\partial z} = \left\{ \dfrac{1}{r}\dfrac{\partial}{\partial r}(r\dfrac{\partial \overline{w}}{\partial r}) \right\} \\ \dfrac{\partial \overline{u}}{\partial r} + \dfrac{\partial \overline{w}}{\partial z} + \dfrac{\overline{u}}{r} = 0 \end{array} \right. \quad (9)$$

The 1st order terms

$$\left\{ \begin{array}{l} \overline{u}\dfrac{\partial \overline{u}}{\partial r} + \overline{w}\dfrac{\partial \overline{u}}{\partial z} = 0 \\ \overline{u}\dfrac{\partial \overline{w}}{\partial r} + \overline{w}\dfrac{\partial \overline{w}}{\partial z} = 0 \end{array} \right. \quad (10)$$

The 2nd order terms

$$\left\{ \begin{array}{l} \dfrac{\partial^2 \overline{u}}{\partial z^2} = 0 \\ \dfrac{\partial^2 \overline{w}}{\partial z^2} - \dfrac{\partial \overline{p}}{\partial z} = 0 \end{array} \right. \quad (11)$$

At this level, we look for the linearized solutions. So, the 1st order terms are neglected. Under this assumption, the analytical solution of the pressure is obtained.

$$\left\{ \begin{array}{l} \overline{p} = \left(\begin{array}{l} -\dfrac{A_1}{2}.\sqrt{2}.\sqrt{-I.\omega}J_0\left(\dfrac{1}{2}.\sqrt{2}.\sqrt{-I.\omega}\,r\right).z \\ +A_2.\sqrt{-I.\omega}J_0\left(\sqrt{-I.\omega}\,r\right) \end{array} \right)e^{-I\omega t} \\ \\ A_1 = \dfrac{I.\omega.A_2\left[I.\gamma_f.\omega\sqrt{-I.\omega}J_0\left(\sqrt{-I.\omega}\right) + J_1\left(\dfrac{1}{2}\sqrt{-I.\omega}\right)\right]}{J_1\left(\dfrac{1}{2}.\sqrt{2}.\sqrt{-I.\omega}\right)} \end{array} \right.$$

$$(12)$$

where, the J_0 and J_1 are the Bessel functions and ω is the dimensionless fluid frequency. A_1 and A_2 are the complex numbers with I being the imaginary unit.

As regards the resolution of the dynamic equilibrium wall, let us resolve the equations in system (6). In order to do this, we take up the linearization process around the initial equilibrium state. We introduce the linearized parameters ($\beta_2 \quad 1$):

$$\overline{P}_3^* = \beta_2^{1/3}\overline{P}_3^0; \qquad \overline{R} = 1 + \beta_2^{1/3}\overline{R}_1 \quad (13)$$

Inserting system (13) into system (6), the approached solution is formulated at the 0th order of $\beta_2^{1/3}$ by:

$$\gamma_f^0 \overline{P}_3^0\big|_{r=1} + \overline{R}_1 = 0 \quad (14)$$

This relation analytically presents the linear correlated to fluid flow pressure to the wall

deformation. This is in totally agreement with many numerical and experimental models that have been reported by Sochi et al. [12] and Leibinger et al. [13].

4. Application and Interpretations

In order to investigate the dynamical behaviors of a three-dimensional flexible tube due to fluid-structure interaction, the geometrical and numerical parameters of the simulation are listed in Table 1.

With the above parameters, the asymptotic restriction (3) and the asymptotic constraints (5) are perfectly verified $\left(S_t = 0.025, \Re_e = 188, \varepsilon = 0.018\right)$. Moreover, the resolution provides a relationship between the fluid-structure characteristics and dimensionless fluid frequency (Fig. 2).

Fig. 3 illustrates the elastic deformation, which provides an assessment the degree of swelling of a rubber tube established by the fluid pressure. These results will be beneficial for a good control of the prevention of an eventual fatigue damage, also that is widely used to prevent or minimize the transmission of dynamic oscillations to a supporting structure that Sommer [14] has revealed in troubleshooting rubber problems. In addition, we observe any stress overtaking of elastic limit and slight displacement of the wall.

With this Buckling factor, Fig. 4 indicates the elastic pressure wave's propagation through the wall in the vicinity at the exit. The validity of solution is realized by Morgan et al. [15].

Table 1   Geometrical and numerical parameters.

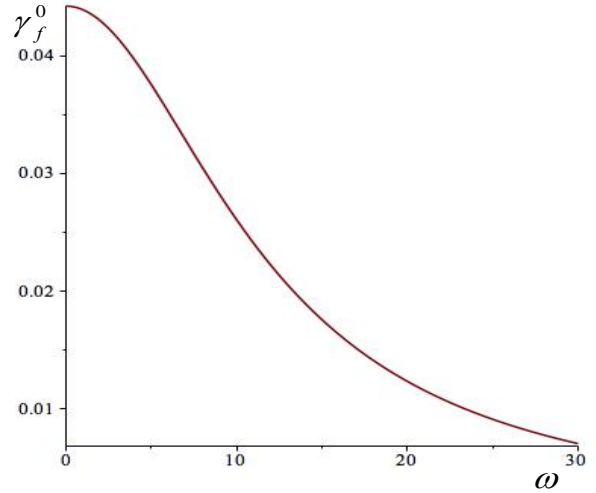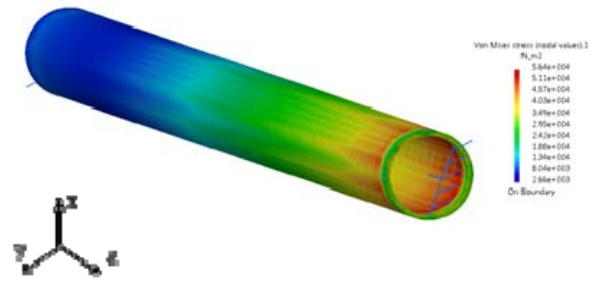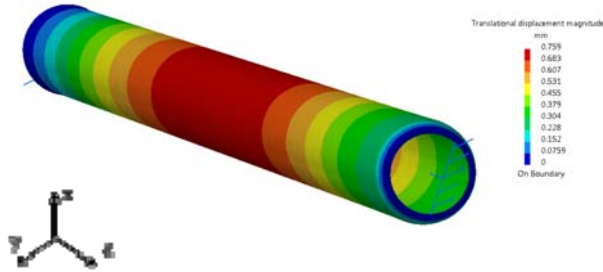| Parameter	Value [Unit]
Fluid (SAE 50W):	
Density	902 [kg·m^{-3}]
Dynamic viscosity	0.86 [Pa·s^{-1}]
Strouhal number (Tref = 0.05 s)	0. 025
Tube (Rubber):	
Density	990 [kg·m^{-3}]
Young's modulus	10^7 [Pa·s]
Poisson's ratio	0.4
Length	0.8 [m]
Radius (at rest)	1.5 [cm]
Thickness	2 [mm]

Fig. 2   System characteristics versus fluid frequency.

(a) Stress contour plot at t' = 0.0125 s (t = 0.25)

(b) Displacement of the tube wall at t' = 0.0125 s (t = 0.25)

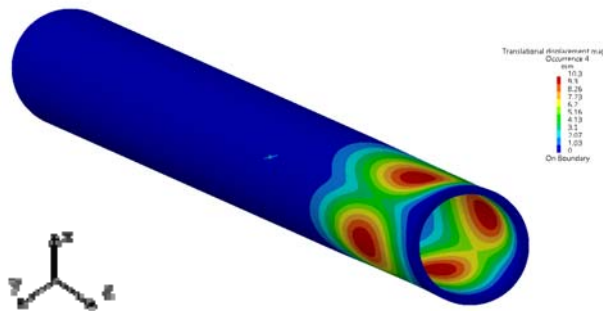**Fig. 3   Three-dimensional tube behavior at Fluid frequency F_{Fluid} = 20 Hz.**

Fig. 4   Translational displacement in buckling analysis.
(Buckling factor = 0.361, t' = 0.0125 and F_{Fluid} = 20 Hz.)

5. Conclusions

In this paper we represented analytically the solution of fluid structure interaction for a compressible fluid flowing through a deformable tube. Similarity solution is developed by Flaherty et al. [16] and Theresa et al. [17]. According to these findings, the dynamic behavior of a flexible tube used in industrial hydraulic systems has been performed.

As future work, the tube wall behavior will be processed considering the axial velocity with a large interval frequency. This process will allow us to determine the rate limit of wall radial displacement in order to validate the asymptotic approach of the model as small deformations.

References

[1] Mehdari, A., Hasnaoui, M., and Agouzoul, M. 2015. "Modelling of Flow through an Elastic Tube with a Variable Radius." In *Proceedings of 12eme Congrès de Mécanique*, Casablanca, Maroc, 21-4.

[2] Grotberg, J. B., and Jensen, O. E. 2004. "Biofluid Mechanics in Flexible Tubes." *Journal of Annual Review of Fluid Mechanics* 36: 121-47.

[3] Squires, T. M., and Quake, S. R. 2005. "Microfluidics: Fluid Physics at the Nano Liter Scale." *Journal of Reviews of Modern Physics* 77: 977-1026.

[4] Eggert, M. D., and Kumar, S. 2004. "Observations of Instability, Hysteresis, and Oscillation in Low-Reynolds Number Flow past Polymer Gels." *Journal of Colloid and Interface Science* 274: 234-42.

[5] Babarit, A., Gendron, B., Singh, J., Melis, C., and Jean, P. 2012. "Modélisation numÉrique et expÉrimentale d'un Systeme Houlomoteur Électro-Actif Déformable." In *Proceedings of 13eme Journées de l'hydrodynanique*, Chatou, France.

[6] Matin, A. 2013. "Main Gas Pipelines: Fracture Resistance Assessment of Pipes." *Journal of Mechanics Engineering and Automation* 3: 127-40.

[7] Gilmanova, A., Bao Le, T., and Sotiroupolos, F. 2015. "A Numerical Approach for Simulating Fluid Structure Interaction of Flexible Thin Shells." *Journal of Computational Physics* 300: 814-43.

[8] Greenshields, C. J., and Weller, H. G. 2005. "A Unified Formulation for Continuum Mechanics Applied to Fluid-Structure Interaction in Flexible Tubes." *International Journal for Numerical Methods in Engineering* 64: 1575-93.

[9] Anliker, M., and Raman, K. R. 1966. "Korotkoff Sounds at Diastole: A Phenomenon of Dynamic Instability of Fluid-Filled Shells." *International Journal of Solids and Structures* 2: 467-91.

[10] Frey, F., and Studer, M. A. 2003. *Analyse des Structures et Milieu Continus: Coques*. Lausane, Suisse: Presses polytechniques et universitaires romandes.

[11] Callegari, A. J., and Reiss, E. L. 1968. "Non-linear Boundary Value Problems for the Circular Membrane." *Archive for Rational Mechanics and Analysis* 31: 390-400.

[12] Sochi, T. 2014. "The Flow of Newtonian and Power Law Fluids in Elastic Tubes." *International Journal of Non-linear Mechanics* 67: 245-50.

[13] Leibinger, J., Dumbser, M., Iben, U., and Wayand, I. 2016. "A Path-Conservative Osher-Type Scheme for Axially Symmetric Compressible Flows in Flexible Visco-Elastic Tubes." *Journal of Applied Numerical Mathematics* 105: 47-63.

[14] Sommer, J. G. 2013. *Trouble Shooting Rubber Problems*. Munich: Hanser Publishers, ISBN 978-1-56990-554-8.

[15] Morgan, G. W., and Ferrante, W. R. 1955. "Wave Propagation in Elastic Tubes Filled with Streaming Liquid." *Journal of Acoustical Society of America* 27: 715-25.

[16] Flaherty, J. E., Keller, J. B., and Rubinow, S. I. 1972. "Post Buckling Behavior of Elastic Tubes and Rings with Opposite Sides in Contact." *Journal of Applied Mathematics* 23: 446-55.

[17] Theresa, B., Verneuil, E., and Hosoi, A. E. 2014. "Elastocapillary Flows in Flexible Tubes: Physics of fluids." *AIP Physics of Fluids* 26: 122103. http://dx.doi.org/10.1063/1.4902509.