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Nonlinear and nonlocal evolution equations driven by Lévy diffusions

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Abstract: We will discuss the interplay between the nonlinear and nonlocal components of the evolution equations. In the particular case of supercritical multifractal conservation laws (CL) the asymptotic behavior, as $t \to \infty$, is dictated by the linearized case. For $\alpha < 1$, the equations driven by infinitesimal generators of Levy α-stable diffusions the solution exhibit shocks (for bounded, odd, and convex on R^+, initial data) which disappear over a finite time. For Lévy α-Linnik diffusions, $0 < \alpha < 2$, the local behavior is strikingly different. The relevant CLs display shocks that do not dissipate over time. Asymptotic explicit solutions for some multifractal CLs are also presented.

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1. Introduction: nonlocal conservation laws. Fractal Burgers equations of the form

$$u_t(t,x) = \mathcal{D}^\alpha u(t,x) - \nabla \cdot g(u(t,x)), \qquad t>0, x\in R^d,) < \alpha \le 2 \tag{1}$$

where the fractional Laplacian D^α is defined as the Fourier multiplier operator $D^\alpha u = \mathcal{F}^{-1}(-|\xi|^\alpha)\mathcal{F}u$, have been introduced in Biler, Funaki and Woyczynski (xxx). The operator is nonlocal and can be rewritten as in integral operator with a singular kernel. The equations describe a variety of physical phenomena. In particular, for or $\alpha = 2$, and $g(u) = |u|^2$, Zeldovich et al. (1982) proposed (1), in combination with the continuity equation, as a model of the Large Scale Structure of the Universe (see, Fig.1). It was formalized in Molchanov, Surgailis and Woyczynski (1997), and Slobodrian (2005) suggested extension of the Large Scale Structure models to the fractal context. .

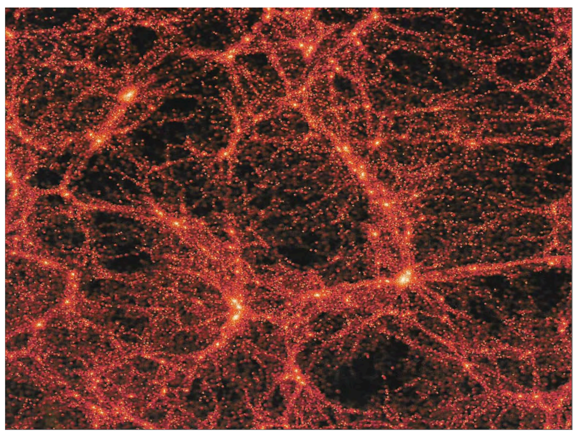

Figure 1: *A simulation of the Large Scale Structure of the Universe using Zeldovich's model.*

In this presentation we will concentrate on the power nonlinearities of the form,

$$g(u) = b\cdot u|u|^{r-1}, \qquad r>1, b\in R^d. \tag{2}$$

2. Critical nonlinearity exponent. The behavior of the solutions of (1) with nonlinearities of the form (2) depends strongly on the relationship between the fractional parameter α and the power r of the nonlinearity (see, Biler, Karch , and Woyczynski (2001). If $r > r_\alpha$, where

$$r_\alpha = 1 + \frac{\alpha-1}{n}, \qquad \alpha > 1,$$

is the critical nonlinearity exponent, we have a weakly nonlinear case, asymptotically behaving linearly:

$$\|u(t) - e^{t\mathcal{D}^\alpha}u(0)\|_{L_p} \longrightarrow 0, \qquad t\to\infty$$

where $e^{t\mathcal{D}^\alpha}$ is the semigroup of operators with D_α as its infinitesimal generator.

In the critical case, $r = r_\alpha$, nonlinear and diffusive terms are balanced, and there exists a specific selfsimilar source solution,

$$U(x,t) = t^{-n/\alpha} U(xt^{-1/\alpha}, 1)$$

such that, asymptically, for any solution u,

$$\|u(t) - U(t)\|_{L_p} \longrightarrow 0, \qquad t \to \infty, \qquad p \geq 1.$$

3. Interacting particle approximation. In the case of quadratic nonlinearity (r=2) Jourdain, Mélèard, Woyczynski (2005), have demonstrated that the equation (1) can be viewed as a hydrodynamic limit of an system of interacting particle driven by the fractional diffusion. More precisely, consider a system $\{X_i^n(t) \in R, i = 1, \dots, n\}$, of initially statistically independent particles, with initial probability distribution $u_0(x)dx$, with their dynamic described, for $\epsilon > 0$, by the system of stochastic differential equations

$$dX_i^n(t) = dS_i^\alpha(t) + \frac{1}{2n} \sum_{j \neq i} \delta_\epsilon \Big(X_i^n(t) - X_j^n(t) \Big)\, dt, \quad i = 1, \dots, n$$

where $S_i^\alpha(t) \in S\alpha S, 1 < \alpha < 2$, are independent α-stable diffusions with independent and stationary increments, and $\delta_\epsilon(x)$ is a regularized Dirac delta $\delta(x)$.

It turns out that the empirical distributions (measure-valued) of the particle positions,

$$\bar{X}^n(t) \equiv \frac{1}{n} \sum_{i=1}^{n} \delta(X_i^n(t))$$

converge weakly, as $n \to \infty$, in probability, to a deterministic measure, with density $u_\epsilon(x,t)\, dx$, solving the fractional Burgers equations with the quadratic nonlinearity,

$$(u_\epsilon)_t = \mathcal{D}^\alpha u - \nabla \Big((\delta_\epsilon * u_\epsilon) \cdot u_\epsilon \Big), \quad u_\epsilon(x,0) = u_0(x).$$

4. The importance of α-stable vs. α-Linnik probabability distributions. In the above introductory sections we concentrated on the diffusions with α-stable distributions but there are other nonlocal fractional diffusions, such as Linnik distributed diffusions, that are also important, and we'll discuss nonlocal and nonlinear evolution equations driven by them as well. Here is a brief comparison of the α-stable and Linnik probability distributions:

For the symmetric α-stable random variable, S, the characteristic function is of the form,

$$\phi_S(\xi; \alpha, c) = \mathbf{E} e^{i\xi S} = \exp(-|c\,\xi|^\alpha), \qquad c > 0, \qquad 0 < \alpha \leq 2.$$

For the symmetric α-Linnik random variable, L,

$$\phi_L(\xi; \alpha, \gamma) = \mathbf{E}e^{i\xi L} = \frac{1}{1 + |\gamma\xi|^{\alpha}}, \qquad \gamma > 0, \qquad 0 < \alpha \leq 2.$$

For $\alpha < 2$, both probability distribution display fat, power-type tails. In particular, in the case of the α-Linnik probability density function (PDF) we have the asymptotic formula,

$$f_L(x; \alpha, 1) \sim \frac{1}{\pi} \left\{ \Gamma(1 + \alpha \sin\left(\frac{\pi\alpha}{2}\right) \right\} |x|^{-(1+\alpha)}, \ as \ x \to \infty.$$

Figure 2 shows the graphs of α-stable, and α-Linnik PDF for different values of parameter α. Note that for $\alpha < 1$, the α-Linnik PDFs display singularities at the origiin. This is clearly illustrated in the third graph showing the cumulative distribution function (CDF) for α-Linnik distributions.

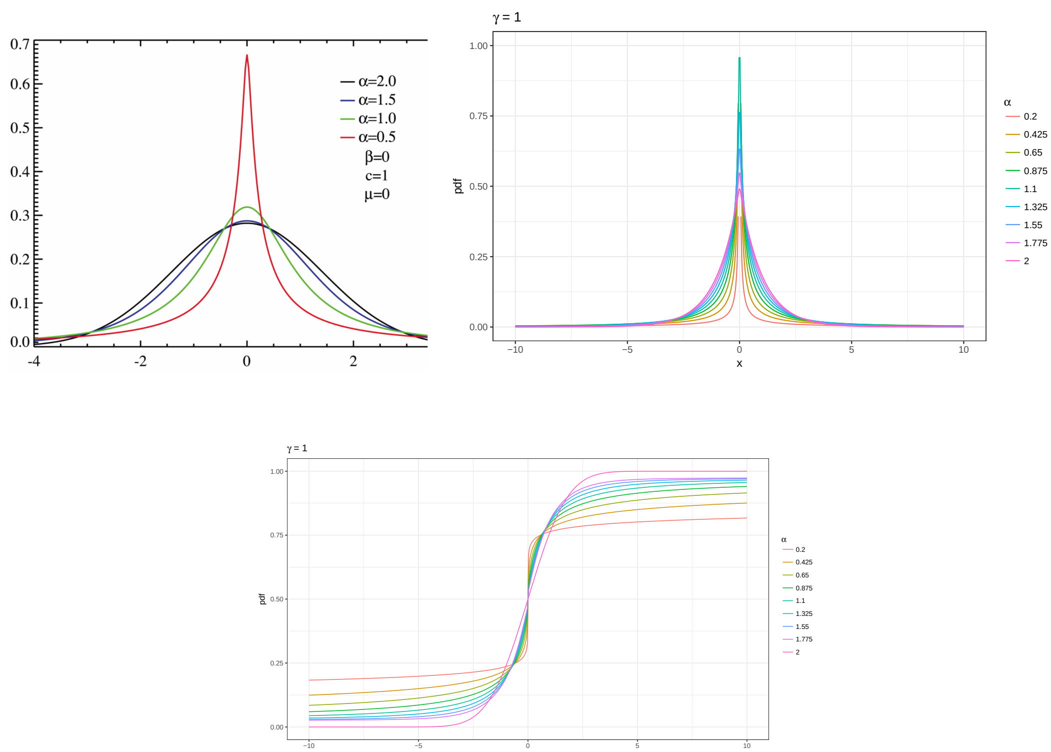

Figure 2: *Stable (top left) and Linnik (top right) PDFs and Linnik CDFs (bottom).*

The fundamental importance of the above distribution types is illustrated by the following two Central Limit Theorems:

Theorem 1. *If* $X_1, X_2, \ldots$ *are independent and identically distributed random variables then* $S_n = X_1 + X_2 + \cdots + X_n$ *converge, as* $n \to \infty$, *in distribution (after some rescaling) to an* α*-stable distribution for some* $\alpha \in (0, 2]$.

Related to the "stability" property is the fact that if $X_1, X_2, \ldots, X_n$ are symmetric α-stable themselves then $Y = n^{-1/\alpha}(X_1 + X_2 + \cdots + X_n)$ has the same distribution as each of the $X_i's$.

But in the case of random summation of random variables the Central Limit Theorem leads to the α-Linnik distributions:

Theorem 2. *If $X, X_1, X_2, \ldots$ are independent and identically distributed random variables and N is an independent of $X_1, X_2, \ldots$ random variable with the geometric distribution, that is,*

$$\mathbf{P}(N = n) = p(1-p)^{n-1}, \qquad n = 1, 2, \ldots, 0 < p < 1,$$

then the following two statements are equivalent:

(i) X is α-stable with respect to geometric summation, i.e.,

$$p^{1/\alpha} \sum_{i=1}^{N} X_i \overset{d}{=} X,$$

(ii) X has the α-Linnik distribution.

5. 1-D Conservation laws driven by multifractal α-stable and α-Linnik diffusions. Mathematical conservation laws are integro-differential evolution equations, such as Navier-Stokes and Burgers equations, expressing the physical principles of conservation of mass, energy, momentum, enstrophy, etc. We'll consider equations of the form,

$$u_t + \mathcal{L}u + (f(u))_x = 0, \qquad u = u(t,x), \quad u(0,x) = u_0(x), \tag{3}$$

where $\mathcal{L}$ is an infinitesimal generator of the semigroup associated with a Lévy process (see, e.g. Samorodnitsky and Taqqu (1995)), and $f : \mathbb{R} \mapsto \mathbb{R}$ is a (nonlinear) function. Such equations are often called *fractal, or anomalous conservation laws.*

Operators $\mathcal{L} = \lim_{h\to 0} (P_h - P_0)/h$

$$P_t f(x) = E^x(f(X(t)) = \int_{\mathbb{R}} f(x+y)\, P(X(t) \in dy)$$

are best described via their actions in the Fourier domain (Fourier multipliers). For a general Lévy processes X_t, we have the identity

$$\mathcal{F}(\mathcal{L}f)(\xi) = -\psi(\xi)\mathcal{F}f(\xi). \tag{4}$$

The "densities' $v(t,x) = P_t f(x)$ satisfies the (generalized) Fokker-Planck evolution equation

$$v_t = \mathcal{L}v.$$

In the case of the usual Brownian motion the infinitesimal operator $\mathcal{L}$ is just the classical Laplacian Δ with the multiplier $-|\xi|^2$.

For the α-stable process with $X(1) \sim S(\alpha, c)$,

$$\mathcal{F}(\mathcal{L}f)(\xi) = -|c\xi|^{\alpha}\mathcal{F}f(\xi), \tag{5}$$

and the infinitesimal generator i $\mathcal{L}$ is called the *fractional Laplacian.*

And, for the α-Linnik process, $X(1) \sim L(\alpha, \gamma)$,

$$\mathcal{F}(\mathcal{L}f)(\xi) = -\log(1 + |\gamma\xi|^{\alpha})(\mathcal{F}f(\xi), \tag{6}$$

By solution to the conservation law

$$u_t + \mathcal{L}u + (f(u))_x = 0, \qquad u = u(t, x),$$

we mean a *mild* solution of the integral equation,

$$u(t) = e^{-t\mathcal{L}}u_0 - \int_0^t \nabla \cdot e^{-(t-\tau)\mathcal{L}} f(u)(\tau)\, d\tau,$$

motivated by the classical Duhamel formula. The existence , uniqueness and other properties related to the probabilistic properties (propagation of chaos)can be found in e.g. , Biler, Karch, and Woyczynski (2001) , Karch and Woyczynski (2008), and Jourdain, Méléard, and Woyczynski (2012).

Multifractal stable operator is defined as follows:

$$\mathcal{L} = -a_0\Delta + \sum_{j=1}^{k} a_j(-\Delta)^{\alpha_j/2}, \tag{7}$$

$0 < \alpha_j < 2, \quad a_j > 0, j = 0, 1, \ldots, k$, where

$$((-\Delta)^{\alpha/2}v) = \mathcal{F}^{-1}(|\xi|^{\alpha}(\mathcal{F}v)(\xi)). \tag{8}$$

Similarly, the *multifractal Linnik operator* will be understood here as the operator of the form

$$\mathcal{L} = -a_0\Delta + \sum_{j=1}^{k} a_j L_{\alpha_j}, \tag{9}$$

where $L_{\alpha}v = \mathcal{F}^{-1}(\log(1 + |\xi|^{\alpha})(\mathcal{F}v)(\xi))$. Note that the parabolic regularization was included in both operators. The following results has been demonstrated in Gunaratnam and Woyczynski (2015).

Theorem 3. *If u is a solution of the multifractal stable, or Linnik, Cauchy problem (3) with $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and $f \in C^1$, is supercritical, i.e., $\limsup_{s\to 0} |f(s)|/|s|^r < \infty$, for some*

$$r > \max((\min(\alpha_1, \ldots, \alpha_k), 1)), \tag{10}$$

then, for every $1 \le p \le \infty$, the relation

$$\lim_{t\to\infty} t^{(1-1/p)/\alpha} \|u(t) - e^{-t\mathcal{L}} u_0\|_p = 0 \tag{11}$$

holds. Moreover,

$$\left\| e^{t\mathcal{L}} * u_0 - \left(\int_{\mathbb{R}} u_0(x)\, dx \right) \cdot p_{\mathcal{L}}(t) \right\|_p \le t^{-(1-1/p)/\alpha} \eta(t),$$

where $p_{\mathcal{L}}(t)$ is the kernel of the operator $\mathcal{L}$ in (3).

For the critical case the situation is different !!! The first order asymptotics of solutions to the Cauchy problem for the Burgers equation is :

$$t^{(1-1/p)/2} \|u(t) - U_M(t)\|_p \to 0, \qquad \text{as} \qquad t \to \infty,$$

where

$$U_M(t,x) = t^{-1/2} e^{-x^2/4t} \left(K(M) + \frac{1}{2} \int_0^{x/2\sqrt{t}} e^{-\xi^2/4}\, d\xi \right)^{-1}$$

is the, so-called, (selfsimilar !!!) source solution such that $u(0,x) = M\delta_0$. The long time behavior of solutions to the classical Burgers equation is genuinely nonlinear (not determined by the asymptotics of its linearization) It is due to precisely matched balancing influence of the regularizing Laplacian and the gradient-steepening quadratic inertial nonlinearity).

Theorem 4. *Let u be a solution of the Cauchy problem (3) with $\mathcal{L} = (-\Delta)^{\alpha/2} + \mathcal{K}$, $1 < \alpha < 2$, and $\mathcal{K}$ being an infinitesimal generator of a Lévy process whose symbol k satisfies*

$$\lim_{\xi\to 0} k(\xi)/|\xi|^\alpha = 0, \tag{12}$$

and $u_0 \in L^1(\mathbb{R}^n)$, $\int_{\mathbb{R}^d} u_0(x)\, dx = M > 0$. Assume

$$\lim_{s\to 0} \frac{f(s)}{s|s|^{(\alpha-1)/n}} \in \mathbb{R}. \tag{13}$$

Then, for each $1 \le p \le \infty$,

$$\lim_{t\to\infty} t^{n(1-1/p)/\alpha} \|u(t) - U(t)\|_p = 0, \tag{14}$$

where $U = U_M$ is the unique selfsimilar solution of the problem (3) with the initial data $M\delta_0$; $U(t,x) = t^{-n/\alpha} U(1, xt^{-1/\alpha})$.

For multifractal critical conservation laws we have the following result:

Theorem 5. *All the statements of the above Theorem 4 are valid for conservation laws driven by multifractal stable, and multifractal Linnik diffusions, with the infinitesimal generator with the symbol*

$$\psi(\xi) = \sum_{j=1}^{n} a_j |\xi|^{\alpha_j}, \qquad a_j > 0, \quad j = 1, 2, \ldots, n,$$

in the stable case,

$$\psi(\xi) = \sum_{j=1}^{n} a_j \log(1 + |\xi|^{\alpha_j}), \qquad a_j > 0, \quad j = 1, 2, \ldots, n,$$

in the Linnik case, and $\alpha = \alpha_* \equiv \min(\alpha_1, \ldots, \alpha_k)$.

6. Shock creation, persistence and dissolution for α-stable and α-Linnik Burgers equation: numerical results. In the case of a quadratic nonlinearity, i.e., for the 1-D fractal Burgers equation, Alibaud, Droniou, Vovelle (2007) proved that the solution of the equation (3) can exhibit shocks (i.e., jump discontinuities) for bounded, odd on $\mathbb{R}$, and convex on $\mathbb{R}^+$ initial data when $\alpha < 1$. No such effect is present in the case $\alpha > 1$, as in that case the fractional Laplacian has a regularizing effect. More precisely. If $\alpha \in (0, 1)$ then , locally in time, shocks in the initial data are preserved, and with continuous initial data, shocks do appear, also locally in time, if the initial data and its derivative are simultaneously large; otherwise no shocks are created.

We will consider three types of odd, decreasing, and convex on the positive half-line initial conditions:

(i) Riemann-type initial data:

$$u_0(x) = \begin{cases} 1, & \text{for} \quad x \leq 0; \\ -1, & \text{for} \quad x > 0, \end{cases}$$

(ii) Piecewise linear (but continuous) data:

$$u_0(x) = \min(1, \max(-10x, -1)),$$

(iii) Smooth, infinitely differentiable data:

$$u_0(x) = (-2/\pi) \arctan(x).$$

Figures 3-9 show the results of numerical experiements for both, stable and Linnik fractional Burgers equations, for different values of parameter α, and different initial data types described above. The behavior of the solutions is strikingly different in the case of stable and Linnik inifinitesimal generators, and they lead to the following formal conjectures.

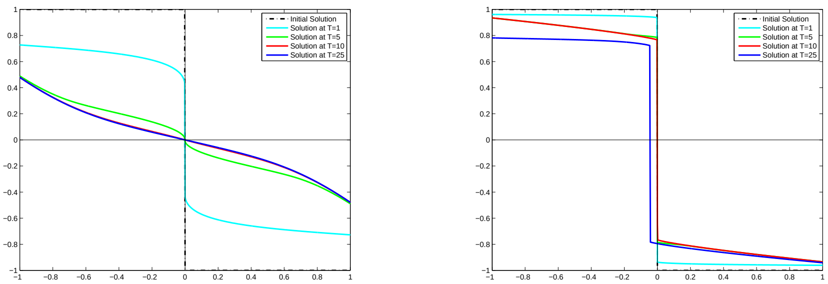

Figure 3. *The solutions of fractional, $\alpha = 0.3$, Burgers equation (quadratic nonlinearity) with Riemann initial condition, at times $t = 1, 5, 10, 25$. Left: stable. Right: Linnik.* ,

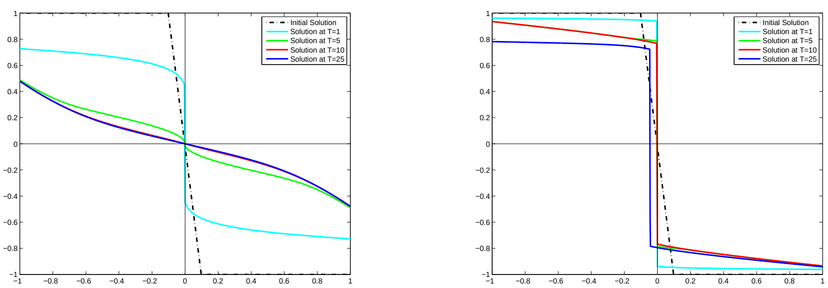

Figure 4. *The solutions of fractional, $\alpha = 0.3$, Burgers equation (quadratic nonlinearity) with continuous but nondifferentiable initial conditions, at times $t = 1, 5, 10, 25$. Left: stable. Right: Linnik.* ,

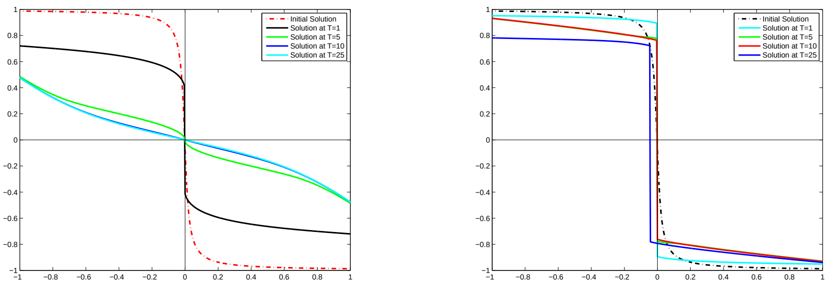

Figure 5. *The solutions of fractional, $\alpha = 0.3$, Burgers equation (quadratic nonlinearity) with smooth infinitely differentiable initial conditions, at times $t = 1, 5, 10, 25$. Left: stable. Right: Linnik.* ,

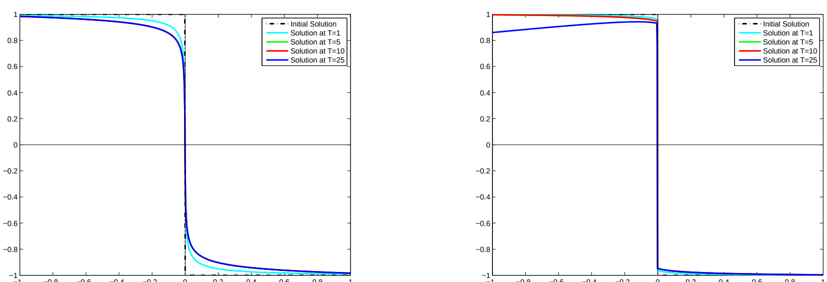

Figure 6. *The solutions of fractional, $\alpha = 1.25$, Burgers equation (quadratic nonlinearity) with Riemann initial condition, at times $t = 1, 5, 10, 25$. Left: stable. Right: Linnik.* ,

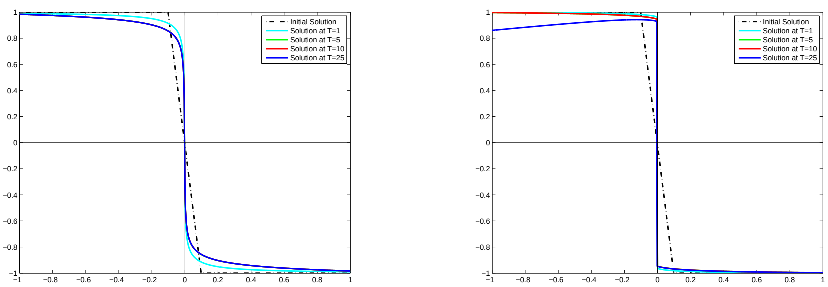

Figure 7. *The solutions of fractional,* $\alpha = 1.25$*, Burgers equation (quadratic nonlinearity) with continuous but nondifferentiable initial condition, at times* $t = 1, 5, 10, 25$*. Left: stable. Right: Linnik.* ,

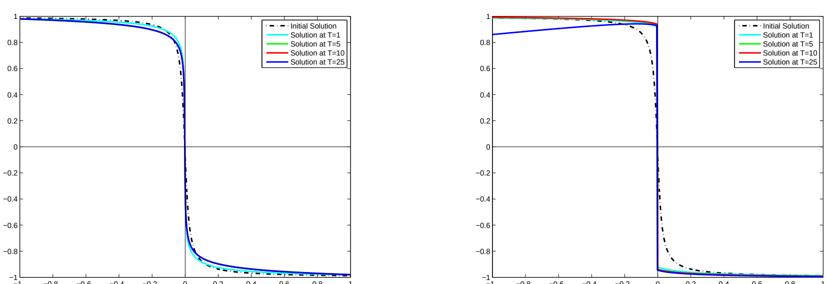

Figure 8. *The solutions of fractional,* $\alpha = 1.25$*, Burgers equation (quadratic nonlinearity) with smooth infinitely differentiable initial condition, at times* $t = 1, 5, 10, 25$*. Left: stable. Right: Linnik.* ,

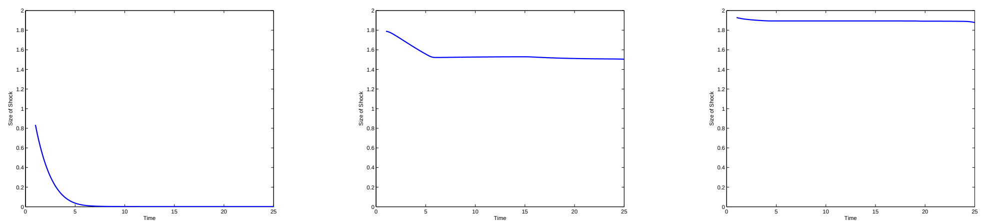

Figure 9. *Smooth initial data: Left: In the* 0.3*-stable case the shock appears at* $T \approx 1$*, and dissolves by* $T \approx 6$*. Middle: In the* 0.3*-Linnik case the shock is again created and its size initially decreases but it stabilizes at* $T \approx 6$*. Right: In the* 1.25*-Linnik case , the shock is again created in finite time, and its size initially decreases, but it stabilizes at* $T \approx 5$*. In the* 1.25*-stable case there are no shocks even for the Riemann initial data.*

Conjecture 1: For an α-stable (fractional) conservation laws with $\alpha < 1$, there exists $t_c > 0$ (obviously greater than the time t_0 of shock creation) such that the solution becomes continuous (and smooth) for all $t > t_c$.

Conjecture 2. For a solution of the α-Linnik conservation law with $\alpha < 1$, there exists a time t_0 such that, at $t > t_0$, the shock is created and its size begins decreasing, but at another time $t_s > t_0$ the size of the shock stops decreasing.

Conjecture 3. For a solution of the α-Linnik conservation law with $\alpha > 1$, there exists a time t_0 such that, at $t > t_0$, the shock is created and its size begins decreasing, but at another time $t_s > t_0$ the size of the shock stops decreasing.

7. Explicit Representations: two-sided case. Given the asymptotical linearized behavior of the solutions of the multifractional conservation laws in the supercritical case described in Theorem 3, it is of importance to find precise information about solutions of the linearized version. In this section we briefly describe an explicit solution of the linear evolution equation driven by the infinitesimal generator for which the Fourier multiplier is of the form:

$$\psi_{(\vec{\alpha};\vec{\beta};\vec{\gamma})}(\omega) = \sum_{j=1}^{n} -\gamma_j |\omega|^{\alpha_j} e^{\frac{i\pi}{2}\beta_j \operatorname{sgn}(\omega)}. \tag{15}$$

$$= \mathcal{F}[v_{(\vec{\alpha};\vec{\beta};\vec{\gamma})}(t,x);\omega]$$

Note that we permit the general asymmetric stable generators in this case. For more results in this direction, see Gorska and Woyczynski (2015). For simplicity's sake , take $\vec{\gamma} = (1, \ldots, 1)$, and a specific uniform partition of the unit interval,

$$\Delta(n,a) = \left\{ \frac{a}{n}, \frac{a+1}{n}, \ldots, \frac{a+n-1}{n} \right\}. \tag{0.1}$$

For the biscale problem the solution has the representation,

$$\mathcal{F}^{-1}[\tilde{v}_{\alpha_1,\beta_1}(t,\omega)\,\tilde{v}_{\alpha_2,\beta_2}(t,\omega);x] \tag{16}$$
$$= H_-(t,-x)\Theta(-x) + H_+(t,x)\Theta(x),$$

where

$$H_+(t,x) = H_+(\alpha_1,\beta_1,\alpha_2,\beta_2;t,x)$$
$$= \frac{Re}{\pi}\int_0^\infty e^{-ix\omega}\exp\left(-t\omega^{\alpha_1}e^{\frac{i\pi}{2}\beta_1} - t\omega^{\alpha_2}e^{\frac{i\pi}{2}\beta_2}\right)d\omega,$$

and

$$H_-(t,x) = H_+(\alpha_1,-\beta_1,\alpha_2,-\beta_2;t,x).$$

The function $\Theta(x)$ is here the usual Heaviside step function.

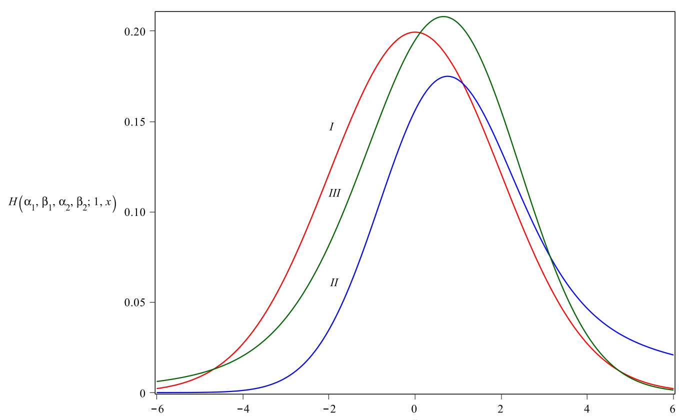

Figure 10. *Multiscale densities* $H(\alpha_1,\beta_1,\alpha_2,\beta_2;t,x)$ *for* $t=1$. *Plot I (red) shows the density* H *for* $\alpha_1=\alpha_2=2$, *and* $\beta_1=\beta_2=0$; *Plot II (blue) corresponds to* $\alpha_1=2$, $\beta_1=0$, $\alpha_2=\frac{1}{2}$, *and* $\beta_2=-\frac{1}{2}$; *Plot III (green) corresponds to* $\alpha_1=2$, $\beta_1=0$, $\alpha_2=\frac{3}{2}$, *and* $\beta_2=-\frac{1}{2}$.

For rational $\alpha_j\in(1,2]$, and β_j, $j=1,2$, such that $\alpha_1=\frac{l}{k}$, $\beta_1=\frac{l-2a}{k}$, $\alpha_2=\frac{p}{q}$, and $\beta_2=\frac{p-2b}{q}$, where l, k, p, q, a, and b, are integers, we have

$$H_+(x,t) = \frac{1}{\pi M}\sum_{r=0}^{\infty}\sum_{j=0}^{lp-1}\frac{(-1)^{r+j}}{r!\,j!}\frac{x^j}{t^{\frac{1+j}{M}+(\frac{m}{M}-1)r}}$$

$$\times\Gamma\big(\frac{1+j}{M}+\frac{m}{M}r\big)\sin\big[\pi u\frac{1+j}{M}-\pi r\big(v-u\frac{m}{M}\big)\big]$$

$$\times {}_{1+m_1}F_{lp}\left(\begin{matrix}1,\Delta(m_1,\frac{1+j}{M}+\frac{m}{M}r)\\ \Delta(lp,1+j)\end{matrix};(-1)^{m_1u+lp}\frac{m_1^{m_1}x^{lp}}{t^{m_1}(lp)^{lp}}\right),$$

where the generalized hypergeometric function F can be represented in terms of the following series,

$$ {}_pF_q\left(\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix};x\right) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n}, $$

where the upper and lower lists of parameters are denoted by (a_p) and (b_q), respectively, and $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol. and where u, and v, are given by $u = \frac{a}{k},\ v = \frac{b}{q},$ for $\alpha_1 > \alpha_2$, and $u = \frac{b}{q},\ v = \frac{a}{k},$ for $\alpha_1 < \alpha_2$. and $m = \min(l/k, p/q)$, $M = \max(l/k, p/q)$, $m_1 = \min(kp, lq), M_1 = \max(kp, lq)$.

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