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# Banach–Saks Property and Property β InCesàro Sequence Spaces

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**Abstract:** A new constant $C$ ($X$) for any Banach space $X$ is introduced. It is proved that $C$ ($X$) < 2 implies the weak Banach–Saks property for the space $X$: In particular, $C(ces_p)$ is found for Cesàro sequence space $ces_p(1 < p < \infty)$. Moreover, it is shown that the space $ces_p(1 < p < \infty)$ has property $(\beta)$.

**Key words:** Banach–Saks property, property $(\beta)$, Cesàro sequence space.

## 1. Introduction

Let $\mathcal{N}$ and $\mathcal{R}$ stand for the set of natural numbers and the set of reals, respectively. Let $(X, \|.\|)$ be a real Banach space and $X^*$the dual space of $X$. By $B(X)$ and $S(X)$, we denote the closed unit ball and the unit sphere of $X$ respectively. For any subset $A$ of $X$ by conv $(A)$ $(\overline{conv}(A))$ we denote the convex hull (closed convex hull) of$A$ $\chi_A$ is the characteristic function of $A.$ Clarkson [1] introduced the concept of uniform convexity.

A norm $\|.\|$ is called UC (uniformly convex) if, for each $\epsilon > 0$, there is $\delta > 0$ such that, for x, y $\in S(X)$, the inequality $\|x - y\| > \epsilon$ implies

$$\left\|\frac{1}{2}(x+y)\right\| < 1 - \delta \ (1)$$

A Banach space $X$ is said to have the Banach–Saks (resp. weak Banach–Saks) property if every bounded (resp. weakly null) sequence $(x_n)$ in $X$ admits a subsequence $(\mathcal{Z}_n)$ such that sequence of its arithmetic means $\{{}^1/_n(\mathcal{Z}_1 + \mathcal{Z}_2 + \cdots + \mathcal{Z}_n)\}$ is convergent in norm (see Ref. [2]).

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It is well known that every Banach space $X$ with the Banach–Saks property is reflexive and the converse is not true (see Ref. [3, 4]) proved that any uniformly convex Banach space $X$ has the Banach–Saks property. Moreover, he also proved that if $X$ is a reflexive Banach space and there is $\theta \in (0,2)$ such that, for every sequence $(x_n)$ in $S(X)$. Weakly convergent to zero, there are $n_1$, $n_2 \in \mathcal{N}$ satisfying
$\|x_{n_1} + x_{n_2}\| < \theta$ . Then $X$ has the Banach–Saks property.

For a sequenc$(x_n) \subset X$ we define
$A((x_n)) = \lim_{n\to\infty} inf\,\{\|x_i + x_j\|: i, j \geq n, i \neq j\}$.

According to Kakutani's result (see Ref. [4]), we introduce the following new geometric constant connected with packing constant (see Ref. [5]) and with the Banach–Saks property:

$$C(X) = \sup\{A((x_n)): x_n \text{ is a weakly null sequence in } S(X)\}$$

Recall that asequence $(x_n)$ is said to be an $\epsilon$-separated sequence if, for some $\epsilon$>0,

$$\text{SeP}(x_n)= \inf\{\|x_n - x_m\|: n \neq m\} > \epsilon.$$

A Banach space is said to be nearly uniformly convex (NUC) if, for every $\epsilon$ >0, there exists $\delta \in (0,1)$such that, for every sequence$(x_n) \subset B(X)$.

With sep$(x_n) > \epsilon$, we have

$$\text{Conv}(\{x_n\}) \cap (1-\delta)\, B(X) \neq \emptyset$$

According to Ref. [6], for any $x \notin B(X)$, the dropdetermined by $x$ is the set

$$D(x, B(X)) = conv(\{x\} \cup B(X))$$

A Banach space $X$ has the drop property (D) if, for every closed set C disjoint with $B(X)$, there exists an element $x \in C$ such that

$$D(x, B(X)) \cap C = \{x\}.$$

In Ref. [7], Rolewicz proved that if the Banach space $X$ has the drop property, then $X$ is reflexive.

For any subset $C$ of $X$, we denote by $\alpha(C)$ its Kuratowski measure of non-compactness, i.e., the infimum of such $\epsilon$>0 for which there is a covering of $C$ by a finite number of sets of diameter less than $\epsilon$.

Goebel and S. ekowski [8] extended the definition of uniform convexity replacing condition (1) by a condition involving the Kuratowski measure of non-compactness, namely, that a norm $\|.\|$ in a Banach space $X$ is ΔUC (Δ-uniformly convex) if, for any $\epsilon$>0, there is $\delta$>0 such that, for each convex set $E$ contained in the $B(X)$such that $\alpha(E) > \epsilon$, we have

$$\inf\{\|x\|: x \in E\} < 1 - \delta.$$

It is well known that ΔUC coincides with NUC.

Rolewicz [7], studying the relationships between NUC and the drop property, has defined property$(\beta)$. A Banach space $X$ is said to have propert y$(\beta)$ if, for any $\epsilon$>0, there exists $\delta$>0 such that

$$\alpha\left(D(x, B(X)) \backslash B(X)\right) < \epsilon$$

whenever 1<$\|x\|$<1 + $\delta$.

The following result will be very helpful for our considerations (see [9]).

A Banach space $X$ has property $(\beta)$ if and only if, for every $\epsilon$>0, there exists $\delta$>0 such that, for each element $x \in B(X)$ and each sequence$(x_n) in\ B(X)$. With sep$(x_n) \geq \epsilon$, there is an index $k$ such that

$$\left\|\frac{x + x_k}{2}\right\| \leq 1 - \delta.$$

Denoted by $l^0$ the space of all real sequences $x= (x(i))$ and by $\{e_i\}$ the natural basisin $l^0$, given $p \in (1, \infty)$ by a Cesàro sequence space, we mean

$$ces_p =:$$

$$\left\{x \in l^0 : \|x\| = \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\right)^p\right)^{\frac{1}{p}} < \infty\right\}$$

For more details, we refer to Refs. [10-12].

## 2. Results

We start with the following general result:

Theorem 1. Any Banach space X with $C(X) < 2$ has the weak Banach–Saks property

Proof. Take a positive number $\epsilon > 0$ such that $\theta = C(X) + \epsilon < 2$. For any weakly null sequence $(x_n) \subset S(X)$, there exists a subsequence $(x_{n_k})$ of $(x_n)$ such that

$$\left\|x_{n_i} + x_{n_j}\right\| < \theta$$

For any $i \neq j$, now, using Kakutani's result (see Ref. [4]), we conclude that the Banach space $X$ has the weak Banach–Saks property.

We will use the following lemma:

Lemma 1. Let $x_n, x_{n+1} \in ces_p$ . Then for any $\epsilon > 0$ and $L$ >0, there exists δ >0 such that

$$|\|x_n + x_{n+1}\|^p - \|x_n\|^p| < \epsilon$$

When ever

$$\|x_n\|^p \leq L\ and \|x_{n+1}\|^p \leq \delta.$$

Proof. Fix $\epsilon > 0$ and $L > 0$, take $\beta = 2^{-P}L^{-1}\epsilon$ and $\delta = 2^{-P}\beta^{P-1}$ . Then for any $x, y \in ces_p$ with $\|x_n\|^p \leq L$ and $\|x_{n+1}\|^p \leq \delta$ we have

$$\|x_n + x_{n+1}\|^p = \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|x_n(i)+x_{n+1}(i)|\right)^p = \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\left|(1-\beta)x_n(i)+\beta\left(x_n(i)+\frac{x_{n+1}(i)}{\beta}\right)\right|\right)^p \le$$

$$\le \sum_{n=1}^{\infty}(1-\beta)\left(\frac{1}{n}\sum_{i=1}^{n}\left||x_n(i)|+\beta\frac{1}{n}\sum_{i=1}^{n}x_n(i)+\frac{x_{n+1}(i)}{\beta}\right|\right)^p$$

$$\le (1-\beta)\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|x_n(i)|\right)^p + \beta\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\left|x_n(i)+\frac{x_{n+1}(i)}{\beta}\right|\right)^p$$

$$\le \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|x_n(i)|\right)^p + \frac{\beta}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|2x_n(i)|\right)^p + \frac{\beta}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{|2(x_{n+1})(i)|}{\beta}\right)^p$$

$$\le \|x_n\|^p + \frac{\epsilon}{2} + \left(\frac{2}{\beta}\right)^{p-1}\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|(x_{n+1})(i)|\right)^p \le \|x_n\|^p + \epsilon.$$

Replacing $x, y$ by $x + x_{n+1}, -x_{n+1}$ respectively, we also conclude that

$$\|x_n\|^p \le \|x_n + x_{n+1}\|^p + \epsilon$$

Hence

$$|\|x_n + x_{n+1}\|^p - \|x_n\|^p| < \epsilon$$

Theorem 2. The space $ces_p$ has property $(\beta)$.

Proof. Suppose $ces_p$ does not have property $(\beta)$, then there exists $\epsilon_0>0$ such that, for any $\delta \in (0, \epsilon_0/(1+2^{1+p}))$, there is a sequence $\{x_n\} \subset S_{(ces_p)}$ with $sep_{(x_n)} > {\epsilon_0}^{1/P}$ and an element $x_0 \in S_{(ces_p)}$ such that

$$\left\|\frac{x_n + x_0}{2}\right\|^p > 1 - \delta$$

for every $n \in \mathcal{N}$.

Fix $\delta \in (0, \epsilon_0/(1+2^{1+p}))$, first, we will show that

$$\lim_{j\to 0}\sup_k \sum_{n=j+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|x_k(i)|\right)^p \le \frac{2^{p+1}\delta}{2^{p-1}} \ . \quad (2)$$

Otherwise, without loss of generality, we can assume that there exists a sequence $\{j_k\}$ such that $j_k \to \infty$ as $k \to \infty$ and

$$\sum_{n=j_k+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|x_k(i)|\right)^p > \frac{2^{p+1}\delta}{2p-1} \ . \quad (3)$$

For every $k \in \mathcal{N}$, let $\delta_1 > 0$ be a real number corresponding to $\epsilon = \delta$ and $L = 1$ in Lemma 1. By absolute continuity of the norm of $x_0$ there exists a positive integer $n_1$ such that

$$\left\|x_0\chi_{\{n_1, n_1+1, n_1+2,\dots\}}\right\|^p = \sum_{n=n+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|x_0(i)|\right)^p < \delta_1$$

Take $k$ so large that $j_k > n_1$ taking into account Lemma 1, convexity of the function $|.|^p$, and Eq. (3), we have

1- $\delta \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{x_k(i)+x_0(i)}{2}\right|\right)^p = \sum_{n=1}^{n_1}\left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{x_k(i)+x_0(i)}{2}\right|\right)^p + \sum_{n=n_1,+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{x_k(i)+x_0(i)}{2}\right|\right)^p \le \frac{1}{2}\sum_{n=1}^{n_1}\left(\frac{1}{n}\sum_{i=1}^{n}|x_0(i)|\right)^p + \frac{1}{2}\sum_{n=1}^{n_1}\left(\frac{1}{n}\sum_{i=1}^{n}|x_k(i)|\right)^p + \sum_{n=n_1,+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{x_k(i)}{2}\right|\right)^p + \delta \le \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{n_1}\left(\frac{1}{n}\sum_{i=1}^{n}|x_k(i)|\right)^p + \frac{1}{2p}\sum_{n=n_1,+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{x_k(i)}{2}\right|\right)^p + \delta \le \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{n_1}\left(\frac{1}{n}\sum_{i=1}^{n}|x_k(i)|\right)^p - \frac{2p-1}{2p}\sum_{n=n_1,+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{x_k(i)}{2}\right|\right)^p + \delta < 1 - 2\delta + \delta = 1-\delta$

This contradiction proves (2).

Since

$$\left(\frac{1}{n_1}\sum_{i=1}^{n_1}|x_k(i)|\right)^p \le \sum_{n=1}^{n_1}\left(\frac{1}{n_1}\sum_{i=1}^{n}|x_k(i)|\right)^p \le 1,$$

We have $|x_k(i)| \le n_1$ for $k \in \mathcal{N}$ and $i=1,2,\ldots.n_1$. Hence, there are a subsequence $(\mathcal{Z}_n)$of $(x_n)$ and a sequen $(a_n)$ of real numbers such that

$$\lim_{k\to\infty} \mathcal{Z}_k(i) = a_i$$

for $i=1,2,\ldots.n_1$ therefore,

$$\sum_{n=1}^{n_1}\left(\frac{1}{n}\sum_{i=1}^{n}|\mathcal{Z}_k(i)-\mathcal{Z}_m(i)|\right)^p < \delta$$

Forn, msufficiently large. Consequently,

$\|\mathcal{Z}_k - \mathcal{Z}_m\|^p = \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|\mathcal{Z}_k(i)-\mathcal{Z}_m(i)|\right)^p = \sum_{n=1}^{n_1}\left(\frac{1}{n}\sum_{i=1}^{n}|\mathcal{Z}_k(i)-\mathcal{Z}_m(i)|\right)^p + \sum_{n=n_1+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|\mathcal{Z}_k(i)-\mathcal{Z}_m(i)|\right)^p \le \sum_{n=1}^{n_1}\left(\frac{1}{n}\sum_{i=1}^{n}|\mathcal{Z}_k(i)-\mathcal{Z}_m(i)|\right)^p + 2^p(\sum_{n=n_1+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|\mathcal{Z}_k(i)|\right)^p + \sum_{n=n_1+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|\mathcal{Z}_m(i)|\right)^p) \le \delta + 2^{p+1}\delta < \epsilon_0, sep(x_n) \le sep(\mathcal{Z}_n) < {\epsilon_0}^{1/p}$. This contradiction shows that $ces_p$ has property $(\beta)$.

Corollary 1

The spaces $ces_p$ and $(ces_p)^*$ have the Banach–Saks property.

Proof.

It is an immediate consequence of Theorem 1 from Ref. [13].

Theorem 3 $\mathrm{C}(ces_p) = 2^{1/p}$

Proof Denote

$$K=sup\left\{A\big((u_n)\big): u_n = \sum_{i=i_{n-1}+1}^{i_n} u_n(i)\, e_i \in S(ces_p), 0 = i_0 < i_1 < i_2 < \cdots, u_n \to^{\omega} 0\right\}$$

Then $\mathrm{C}(ces_p) \ge K$. Moreover, for any $\epsilon>0$; there is a sequence $(x_n) \subset S(ces_p)$ with $x_n \to^{\omega} 0$ such that

$$A\big((x_n)\big) + \epsilon > C(ces_p)$$

By the definition of $A\big((x_n)\big)$, there exists a subsequence $y_n$.of $x_n$.such that

$$\|y_n + y_m\| + 2\epsilon > C\big(ces_p\big) \tag{4}$$

For any $n, m \in \mathcal{N}$ with $m \neq n$ take $v_1 = y_1$. Then, by the absolute continuity of the norm of $y_1$ there exists $i_1 \in \mathcal{N}$ such that

$$\left\| \sum_{i=i_1+1}^{\infty} v_1(i)e_i \right\| < \epsilon.$$

Putting $\mathcal{Z}_1 = \sum_{i=1}^{i_1} v_1(i)e_i$, we have

$$\|\mathcal{Z}_1 + y_m\| = \left\| y_1 + y_m - \sum_{i=i_1+1}^{\infty} v_1(i)e_i \right\| \geq \|y_1 + y_m\| - \epsilon$$

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For any $m > 1$ hence by Eq. (4), we have

$$\|\mathcal{Z}_1 + y_m\| + 3\epsilon > C\big(ces_p\big)$$

For any $m > 1$, Since $y_n(i) \to 0$ for $i$=1, 2, …., there exists $n_2 \in \mathcal{N}$ with $n_2 > n_1$ such that

$$\left\| \sum_{i=1}^{i_1} y_n(i)e_i \right\| < \epsilon$$

Whenever $n \geq n_2$, define $v_2 = y_{n_2}$, then there is $i_2 > i_1$ such that

$$\left\| \sum_{i=i_2+1}^{\infty} v_2(i)e_i \right\| < \epsilon\,.$$

Taking $\mathcal{Z}_2 = \sum_{i=i_1+1}^{i_2} v_2(i)e_i$, we obtain

$$\|\mathcal{Z}_1 + \mathcal{Z}_2\| = \left\| y_1 - \sum_{i=i_1+1}^{\infty} v_1(i)e_i + y_{n_2} - \sum_{i=1}^{i_1} v_2(i)e_i - \sum_{i=i_2+1}^{\infty} v_2(i)e_i \right\| \geq \left\|y_1 + y_{n_2}\right\| - 3\epsilon$$

Hence by eq. (4), we immediately obtain

$$\|\mathcal{Z}_1 + \mathcal{Z}_2\| + 5\epsilon > C\big(ces_p\big)$$

Suppose that increasing sequences $\{i_j\}_{j=1}^{k-1}$,$\{n_j\}_{j=1}^{k-1}$of natural numbers and a sequence $\{\mathcal{Z}_j\}_{j=1}^{k-1}$ of elements of $ces_p$ are already defined and

$$\|\mathcal{Z}_n + \mathcal{Z}_m\| + 6\epsilon > \mathrm{C}\big(\mathrm{ces_p}\big)$$

For$\mathrm{m}, \mathrm{n} \in \{1,2,\dots,\mathrm{k}-1\}$, $\mathrm{m} \neq \mathrm{n}$ Since there exists$y_n(i) \to 0$ for $i = 1,2,\dots$ there exists natural number $n_k > n_{k-1}$ such that

$$\left\| \sum_{i=1}^{i_{k-1}} y_n(i)e_i \right\| < \epsilon$$

Provided $n > n_k$, put $v_k = y_{n_k}$, then there is $i_k > i_{k-1}$ such that

$$\left\| \sum_{i=i_{k+1}}^{\infty} v_k(i)e_i \right\| < \epsilon$$

Defining $\mathcal{Z}_k = \sum_{i=i_{k-1}+1}^{i_k} v_k(i)e_i$, we obtain

$$\|\mathcal{Z}_j + \mathcal{Z}_k\| = \left\| y_{n_j} - \sum_{i=1}^{i_{j-1}} v_j(i)e_i - \sum_{i=i_{j+1}}^{\infty} v_j(i)e_i + y_{n_k} - \sum_{i=1}^{i_{k-1}} v_k(i)e_i - \sum_{i=i_{k+1}}^{\infty} v_k(i)e_i \right\| \geq \left\| y_{n_j} + y_{n_k} \right\| - 4\epsilon$$

For $j = 1,2,\dots,k-1$, hence, by Eq. (4), we obtain

$$\|\mathcal{Z}_j + \mathcal{Z}_k\| + 6\epsilon > C(ces_p)$$

For $j = 1,2,\dots,k-1$ using the induction principle, we can find a sequence $(\mathcal{Z}_n)$ satisfying the following conditions:

(1) $\mathcal{Z}_n = \sum_{i=i_{n-1}+1}^{i_n} v_n(i)e_i$, $where\ 0 = i_0 < i_1 < i_2 < \cdots;$

(2) $\|\mathcal{Z}_n + \mathcal{Z}_m\| + 6\epsilon > C(ces_p)\ for\ m,n \in \mathcal{N}, m \neq n;$

(3) $\|\mathcal{Z}_n\| \leq 1\ for\ n = 1,2,\dots;$

(4) $\mathcal{Z}_n \to^{\omega} 0\ as\ n \to \infty$

Define $u_n = \mathcal{Z}_n/\|\mathcal{Z}_n\|$ for each $n \in \mathcal{N}$, then every $u_n \in S(ces_p)$ and

$$\|u_n + u_m\| = \left\| \frac{z_n}{\|z_n\|} + \frac{z_m}{\|z_m\|} \right\| \geq \|z_n + z_m\| \geq C(ces_p) - 6\epsilon$$

for any $m,n \in \mathcal{N}, m \neq n$, by the arbitrariness of $\epsilon$ we have $C(ces_p) = K$.

Let $\epsilon>0$ be given. Take $n_\epsilon \in \mathcal{N}$ such that

$$\sum_{k=i_{n_\epsilon}+1}^{\infty} \left(\frac{a}{k}\right)^p < \epsilon,$$

where

$$a = \sum_{i=i_{n_{\epsilon-1}}+1}^{i_{n_\epsilon}} |u_{n_\epsilon}(i)|.$$

Hence, for any $m > n_\epsilon$ we have

$$\|\boldsymbol{u}_{n_\epsilon} + \boldsymbol{u}_m\|^p = \sum_{i=i_{n_{\epsilon-1}}+1}^{i_{m-1}} \left(\frac{1}{k}\sum_{i=1}^{k} |u_{n_\epsilon}(i)|\right)^p + \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{k}\sum_{i=1}^{k}\left(a + \sum_{i=1}^{k}|u_m(i)|\right)\right)^p$$

$$\geq \sum_{i=i_{n_{\epsilon-1}}+1}^{i_{m-1}} \left(\frac{1}{k}\sum_{i=1}^{k} |u_{n_\epsilon}(i)|\right)^p + \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{k}\sum_{i=1}^{k} |u_{n_\epsilon}(i)|\right)^p = \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{k}\sum_{i=1}^{k} |u_{n_\epsilon}(i)|\right)^p$$

$$- \sum_{k=i_{n_\epsilon}+1}^{\infty} \left(\frac{a}{k}\right)^p + \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{k}\sum_{i=1}^{k} |u_m(i)|\right)^p > 1 - \epsilon$$

$$+1 = 2 - \epsilon$$

$$A((u_n)) \geq (2-\epsilon)^{1/p}.$$

On the other hand, for $\epsilon$ mentioned above, by Lemma 1, there exists $\delta>0$ such that

$$|\|x+y\|^p - \|x\|^p| < \epsilon$$

Whenever $\|x\|^p \leq 1$ and $\|y\|^p < \delta$, take $n_\delta \in \mathcal{N}$ such that

$$\sum_{k=i_{n_\delta}+1}^{\infty} \left(\frac{a}{k}\right)^p < \delta,$$

$$a = \sum_{i=i_{n_\delta}+1}^{i_{n_\delta}} |u_{n_\delta}(i)|.$$

Hence, for any m $> n_\delta$ we have

$$\|u_{n_\delta} + u_m\|^p = \sum_{i=i_{n_{\delta-1}}+1}^{i_{m-1}} \left(\frac{1}{k}\sum_{i=1}^{k}|u_{n_\delta}(i)|\right)^p + \sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{k}\left(a + \sum_{i=1}^{k}|u_m(i)|\right)\right)^p$$

$$\leq \sum_{k=i_{n-1}+1}^{\infty} \left(\frac{1}{k}\sum_{i=1}^{k}|u_n(i)|\right)^p + \sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{k}\left(a + \sum_{i=1}^{k}|u_m(i)|\right)\right)^p$$

$$= \|u_{n_\delta}\|^p + \sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{k}\left(a + \sum_{i=1}^{k}|u_m(i)|\right)\right)^p - \sum_{i=i_{m-1}+1}^{i_{m-1}} \left(\frac{1}{k}\sum_{i=1}^{k}|u_m(i)|\right)^p + \|u_m\|^p < 2 - \epsilon,$$

$A((u_n)) \geq (2-\epsilon)^{1/p}$ Therefore, by the arbitrariness of $\epsilon$ we obtain $C(ces_p) = 2^{1/p}$ which finishes the proof of the theorem.

Corollary 2. The space $ces_p$ has the Banach–Saks property.

Proof. Note that, for reflexive Banach spaces, the Banach–Saks property is equivalent to the weak Banach–Saks property. Hence, by Theorems 1 and 3, we conclude the thesis of Corollary 2.

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