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Strain Gradient Effects in a Thermo-Elastic Continuum with Nano-Pores

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Abstract: The thermo-mechanical balance equations for a porous material with big irregular pores are derived from the general ones for a medium with ellipsoidal microstructure by imposing the kinematical constraint of micro-stretch bounded to the macro-deformation: in this case the microstructure disappears apparently (it becomes latent) and the response of the material involves higher gradients of the displacement without incurring known constitutive inconsistencies.

Key words: Continua with latent microstructure, higher order continua, porous solid, strain gradient thermo-elasticity.

1. Introduction

In Giovine [1] the general mechanical balance equations for a medium with ellipsoidal microstructure were obtained for the case in which each material element of the body contains a nano-pore filled by an inviscid fluid or an elastic inclusion, both of negligible mass, which could have a microstretch different from, and independent of, the local affine deformation ensuing from the macromotion: then it allows distinct microstrains along principal axes of microdeformation, in absence of microrotations. Therefore, that theory includes, as a particular case, the voids theory of Nunziato et al. [2] which describes continua with "small" spherical pores which only may contract and expand homogeneously: it suffices to constrain the microstretch to be spherical.

The refinement of the Cauchy theory of Giovine [1] was necessary to characterize the more complex structure (see also Bear et al. [3] and Cowin [4]), even if some problems of physical concreteness in Grioli [5] or of mathematical hardness in Cieszko [6] came out. The volume fraction of the pores in Nunziato et al. [2] was not sufficient to describe the microdeformation of the holes when they are large: in fact, the linear voids theory of Cowin et al. [7] does not predict size effects in torsion of bars in an isotropic material, while they occur both in bending and in torsion, as observed by Lakes [8] for bones and polymer foam materials. Instead the linear theory for a medium with ellipsoidal microstructure was used to find solutions of micro-vibrations, plane waves and macro-accelerations waves (see Giovine [9] and §3 of Giovine [10]).

In this paper we suppose that the nano-pores in the material elements are void of matter, and so we obtain the equations of balance for a thermo-elastic porous material with big holes by imposing, as an internal constraint that the microstate is completely determined by the macrostrain, so that only an indirect trace of the microstructure remains: it becomes latent (see Capriz [11]). The form of the Cauchy's equation seems the classical one, but some traditional tenets are abandoned, that is, the stress tensor need not be symmetric and may depend on higher derivatives of displacement such as acceleration gradients, moreover, the energy flux, stress power apart, need not to be equal to the heat flux only. The mechanical case was studied by Giovine et al. [12], where an application to the propagation of asymptotic waves compatible with such a model was

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also considered: physical situations corresponding to an axisymmetric motion either in spherical or cylindrical symmetry were considered, and it was shown that the time evolution of the wave amplitude factor is governed by the spherical and cylindrical Korteweg-deVries equations, respectively.

At the end, the concept of latent microstructure is useful in offering an interpretation of constitutive prescriptions, involving the higher derivatives of displacement, which allows one to circumvent certain apparent inconsistencies with the second law of thermodynamics. An interpretation which is in line with an old remark of Toupin [13] on the possibility of viewing hyper-elastic materials of second grade is as Cosserat's continua whose microrotations are constrained (see also Refs. [14-18]).

2. Balance Equations and Jump Conditions for a Body with Ellipsoidal Microstructure

We identify the continuous material body with ellipsoidal microstructure $\mathcal{B}$ with a fixed homogeneous and free of residual stresses region of the three dimensional Euclidean space $\mathcal{G}$, called the "natural" reference placement $\mathcal{B}_*$ (see, for example, §83 of Truesdell et al. [19]).

We suppose that each material element of the continuum contains a nano-pore which is capable to have a microstretch different from, and independent of, the local affine deformation ensuing from the macromotion, as it is the case when the cavity is filled by an inviscid fluid or an elastic inclusion, both of negligible mass.

Therefore, if we denote the generic material element of $\mathcal{B}_*$ by X, the thermo-mechanical behaviour of $\mathcal{B}$ is described by three smooth mappings on $\mathcal{B}_* \times \mathcal{R}$ ($\mathcal{R}$ is the set of real numbers):

- The spatial position $x \in \mathcal{G}$, at time τ, of the material point which occupied the position X in the reference placement $\mathcal{B}_*$;
- The left Cauchy-Green tensor of the micro-deformation $U \in \mathrm{Sym}^+$, at time τ, of the associated nano-pore (Sym^+ being the collection of second-order symmetric and positive definite tensor fields); and
- The absolute positive temperature $\theta > 0$.

The spatial position $x(X, \tau)$ is a one-to-one correspondence, for each τ, between the reference placement $\mathcal{B}_*$ and the current placement $\mathcal{B}_\tau = x(\mathcal{B}_*, \tau)$ of the body $\mathcal{B}$ and, so, the deformation gradient

$$\mathrm{F} \coloneqq \nabla \mathrm{x}(\mathrm{X},\tau) \left(= \frac{\partial \mathrm{x}}{\partial \mathrm{X}}(\mathrm{X},\tau) \right)$$

is a second order tensor with positive determinant.

Through the inverse mapping $X(x, \tau)$ of x, we can consider all the relevant fields in the theory as defined over the current placement $\mathcal{B}_\tau$ as well as over the reference placement $\mathcal{B}_*$ of the body $\mathcal{B}$.

Hence, in a continuum with ellipsoidal microstructure [1] a rotation $Q = \mathrm{e}^{-\mathcal{E}s}$ of the observer of characteristic vector s, where $\mathcal{E}$ is the third-order Ricci's permutation tensor (skew with respect to all indices) and e the basis of natural logarithms, causes the symmetric tensor U to change into $U_s = QUQ^T$; moreover, the infinitesimal generator $\mathcal{A}$ of the group of rotations on the microstructure in Sym^+ [that is, the operator describing the effect of a rotation of the observer on the value U_s of the microstructure to the first order in s (see §3 of Capriz [20])], is given by

$$\mathcal{A}(\mathrm{U}) \coloneqq \left.\frac{d\mathrm{U}_s}{ds}\right|_{s=0} \tag{1}$$

(in components: $\mathcal{A}_{ijk} = \mathrm{U}_{ijk}\,\mathcal{E}_{ljk} + \mathcal{E}_{ikl}\mathrm{U}_{lj}$); $\mathcal{A}$ is a third-order tensor, symmetric and positive definite in the first two indices, that is $\mathcal{A}c \in \mathrm{Sym}^+$, for all vectors c.

The expression of the kinetic energy density per unit mass of microstructured bodies is the sum of two terms, the classical one $\frac{1}{2}\dot{\mathrm{x}}^2$ due to the translational inertia and the microstructured one $\kappa(\mathrm{U},\dot{\mathrm{U}})$ due to the inertia related to the admissible dilatational micromotions of the pores' boundaries (the superposed dot denotes material time derivative).

This additional term is a non-negative scalar function, homogeneous in U, such that $\kappa(\mathrm{U},0)=0$ and $\frac{\partial^2\kappa}{\partial\dot{\mathrm{U}}^2}\neq 0$, and it is related to the kinetic co-energy density $\chi(\mathrm{U},\dot{\mathrm{U}})$ by the Legendre transform

$$\frac{\partial\chi}{\partial\dot{\mathrm{U}}}\cdot\dot{\mathrm{U}}-\chi=\kappa \tag{2}$$

The kinetic co-energy χ, as κ, must have the same value for all observers at rest, that is, it must be invariant under the Galilean group and hence satisfy the condition

$$\dot{\mathcal{A}}^*\frac{\partial\chi}{\partial\dot{\mathbf{U}}}=-\mathcal{A}^*\frac{\partial\chi}{\partial\mathbf{U}} \tag{3}$$

where, the third-order tensor $\mathcal{A}^*$ is defined through the relation:

$$(\mathcal{A}^*\mathrm{C})\cdot\mathrm{c}=\mathrm{C}\cdot(\mathcal{A}c)$$

for all second-order tensors C and all vectors c.

The use of Eq. (1) into Eq. (3) and the multiplication of both sides by the Ricci's tensor $\mathcal{E}$ gives the following kinematic compatibility relation:

$$\mathrm{skw}\left[\dot{\mathrm{U}}\frac{\partial\chi}{\partial\dot{\mathrm{U}}}+\mathrm{U}\frac{\partial\chi}{\partial\mathrm{U}}\right]=0 \tag{4}$$

where, "skw" denotes the skew part of a second-order tensor: $\mathrm{skw}(\cdot)=2^{-1}\big((\cdot)-(\cdot)^T\big)$ (and "sym" the symmetric one: $\mathrm{sym}(\cdot)=2^{-1}\big((\cdot)+(\cdot)^T\big)$).

All the admissible thermo-kinetic processes for porous solids with large irregular voids are governed by the following general system of balance equations proposed by Giovine [1]; they are the mass conservation, the Cauchy equation, the micro-momentum and moment of momentum balances, the extended Neumann energy equation and the entropy inequality in the Lagrangian description, respectively:

$$\rho_*=\rho\det\mathrm{F} \tag{5}$$

$$\rho_*\ddot{\mathrm{x}}=\rho_*\mathrm{f}+\mathrm{Div\,P} \tag{6}$$

$$\rho_*\left[\frac{d}{d\tau}\left(\frac{\partial\chi}{\partial\dot{\mathrm{U}}}\right)-\frac{\partial\chi}{\partial\mathrm{U}}\right]=\rho_*\mathrm{H}-\mathrm{Y}+\mathrm{Div}\,\Lambda \tag{7}$$

$$\mathcal{E}(\mathrm{PF}^T)=\mathcal{A}^*\,\mathrm{Y}+(\nabla\mathcal{A}^*)\,\Lambda \tag{8}$$

$$\rho_*\dot{\varepsilon}=\mathrm{P}\cdot\dot{\mathrm{F}}+\mathrm{Y}\cdot\dot{\mathrm{U}}+\Lambda\cdot\nabla\dot{\mathrm{U}}+\rho_*\lambda-\mathrm{Div\,k} \tag{9}$$

where, ρ is the mass density and ρ_* its value in the reference placement $\mathcal{B}_*$; Div means the trace of the nabla: Div $(\cdot)$:= tr $\big(\nabla(\cdot)\big)$; f is the vector body force, P the first Piola-Kirchhoff stress tensor, ε the specific internal energy density per unit mass, λ the rate of heat generation due to irradiation or heating supply and k the referential heating flux.

Moreover, on the left hand side of the balance equation of micro-momentum (7) the Lagrangian derivative of the kinetic co-energy χ appears, while, on the right hand side, $\rho_*\mathrm{H}$ and -Y are the resultant second-order symmetric tensor densities of external and internal microactions, respectively: the first one is interpreted as a controlled pore pressure and the other one includes interactive forces between the gross and fine structures as well as internal dissipative contributions due to the stir of the pores' surface. Finally, Λ is the referential microstress third-order tensor, symmetric in the first two indices, which is related to the capability of recognizing boundary microtractions, even if, in some cases, it expresses weakly non-local internal effects due to the impossibility of defining a physically significant connection on the manifold of the microstructural kinetic parameter U (see Capriz et al. [21, 22]).

The balance of moment of momentum in Eq. (8) assumes a more significant expression when we use the representation (1) for $\mathcal{A}$; in fact we have

$$\mathrm{skw}(\mathrm{PF}^T)=2\,\mathrm{skw}\,(\mathrm{UY}+\nabla\mathrm{U}\odot\Lambda) \tag{10}$$

where the tensor product $\odot$ between third-order tensors is so defined:

$$(\nabla\mathrm{U}\odot\Lambda)_{ij}\coloneqq\mathrm{U}_{ih,L}\Lambda_{jhL} \tag{11}$$

The imbalance of entropy still applies in the classical Clausius-Duhem form:

$$\rho_*\theta\dot{\eta}\geq\rho_*\lambda-\mathrm{Div\,k}+\theta^{-1}\,(\mathrm{k}\cdot\nabla\theta) \tag{12}$$

where η is the density of entropy per unit mass; moreover, if we introduce the Helmholtz free energy per unit mass

$$\psi:=\varepsilon-\theta\eta$$

and use in Eq. (9), we obtain a reduced version of this inequality, that is,

$$\rho_*(\dot{\psi}+\eta\dot{\theta}) \leq \mathrm{Y}\cdot\dot{\mathrm{U}}+\mathrm{P}\cdot\dot{\mathrm{F}}+\Lambda\cdot\nabla\dot{\mathrm{U}}-\frac{\mathrm{k}\cdot\nabla\theta}{\theta} \quad (13)$$

At the end, by using the balance of moment of momentum (10), we have an objective version of the inequality which is indifferent to changes in observer:

$$\rho_*(\dot{\psi}+\eta\dot{\theta})+\frac{\mathrm{k}\cdot\nabla\theta}{\theta} \leq \mathrm{Y}\cdot(\dot{\mathrm{U}}+\mathrm{UW}-\mathrm{WU})+$$
$$+\mathrm{sym}\,(\mathrm{PF}^T)\cdot\mathrm{D}+\Lambda\cdot\left[\nabla\dot{\mathrm{U}}-\mathrm{W}\nabla\mathrm{U}+(\nabla\mathrm{U})^t\,\mathrm{W}\right] \quad (14)$$

where $\mathrm{D}\in\mathrm{Sym}$ is the stretching (that is, the symmetric part of the gradient of velocity $\mathrm{L}:=\frac{\partial\dot{\mathrm{x}}}{\partial\mathrm{x}}=\dot{\mathrm{F}}\mathrm{F}^{-1}$) and $\mathrm{W}\in\mathrm{Skw}$ the spin tensor (that is, the skew part), so that L = D + W; in Eq. (14) the minor right transposition means $(\cdot)^t_{ijl}=(\cdot)_{ilj}$, while the minor left one is so defined: ${}^t(\cdot)_{ijl}=(\cdot)_{jil}$.

Remark: We observe that the voids theories of Nunziato et al. [2] and Capriz et al. [23] are immediately recovered when the microstretch *U* is constrained to be spherical (see, also, §5 of Giovine [1]).

In addition to balance Eqs. (5-7) and (9-10), we need the balance equations at a surface of discontinuity, namely a propagating wave Σ. As it is customary, we assume that the smooth movable surface Σ that traverses the body *B*, is oriented and we denote by *n* the unit normal vector to Σ in the reference placement $\mathcal{B}_*$ and by υ_n the corresponding non-zero normal speed of displacement of Σ at point (X, τ) in the reference placement.

We further assume that some field related to the motion of $\mathcal{B}$ (excepting *x*, *U* and *θ*) suffers jump discontinuity across Σ and so we employ the usual notation $[\![\cdot]\!]$ for jumps, so that

$$[\![f]\!]=f^+-f^- \quad (15)$$

where f^+ or f^- refers to the limit of *f* as the wave is approached from the right or left, respectively.

Therefore, we can write classical Kotchine's equations, as modified in order to take into account microstructural effects, and a relation that restricts the jump of micro-momentum (see, also, Capriz et al. [24] and Paoletti [25]) as it follows:

$$[\![\rho_*(\upsilon_n-\dot{\mathrm{x}}\cdot\mathrm{n})]\!]=0 \quad (16)$$
$$[\![\rho_*\upsilon_n\dot{\mathrm{x}}+\mathrm{Pn}]\!]=0 \quad (17)$$
$$\left[\!\!\left[\rho_*\upsilon_n\frac{\partial\kappa}{\partial\dot{\mathrm{U}}}(\mathrm{U},\dot{\mathrm{U}})+\Lambda\mathrm{n}\right]\!\!\right]=0 \quad (18)$$
$$\left[\!\!\left[\rho_*\upsilon_n\left(\varepsilon+\frac{1}{2}\dot{\mathrm{x}}^2+\kappa(\mathrm{U},\dot{\mathrm{U}})\right)\right]\!\!\right]=$$
$$\geq[\![(\mathrm{h}-\mathrm{P}^T\dot{\mathrm{x}}-\Lambda^*\dot{\mathrm{U}})\cdot\mathrm{n}]\!] \quad (19)$$
$$[\![\rho_*\upsilon_n\theta\eta]\!]\geq[\![\mathrm{h}\cdot\mathrm{n}]\!] \quad (20)$$

The form of the jumps across a propagating wave of higher order time derivatives of principal fields can be obtained from the balance Eqs. (5-7) and (9).

3. The Porous Solid as a Thermoelastic Continuum with Strain Gradient Effects

The porous material with empty large interstices, which do not diffuse through the matrix material, is depicted as a continuous body whose microstructure *U* is *internally constrained* to be equal to the left (macro) Cauchy-Green tensor B:

$$\mathrm{U}=\mathrm{B}=\mathrm{FF}^T\ \in\mathrm{Sym}^+ \quad (21)$$

therefore only an indirect trace of the presence of the microstructure remains and one dramatic fact emerges: the microstructure becomes *latent* and the Piola stress tensor may depend on higher derivatives of displacement as well as on acceleration gradients.

Before to see this, we use the constraint (21) and, after, the balance of angular momentum (8) in the expression of the density of total power of internal actions $\mathcal{P}_{int}$ to have the following equivalences:

$$-\mathcal{P}_{int}=\ \mathrm{P}\cdot\dot{\mathrm{F}}+\mathrm{Y}\cdot\dot{\mathrm{B}}+\Lambda\cdot\nabla\dot{\mathrm{B}}=$$
$$=[\mathrm{P}+2\mathrm{YF}+2\Lambda\odot\nabla(\mathrm{F}^T)]\cdot\dot{\mathrm{F}}+2\left[\Lambda\odot\nabla(\dot{\mathrm{F}}^T)\right]\cdot\mathrm{F}$$
$$=$$
$$=[\mathrm{PF}^T+2\mathrm{YB}+2\Lambda\odot\nabla\mathrm{B}]\cdot\mathrm{L}+2[\Lambda\odot\nabla(\mathrm{L}^T)]\cdot\mathrm{B}=$$
$$=\mathrm{sym}\,[\mathrm{PF}^T+2(\mathrm{YB}+\Lambda\odot\nabla\mathrm{B})]\cdot\mathrm{D}+\Omega\cdot\frac{\partial^2\dot{\mathrm{x}}}{\partial\mathrm{x}^2} \quad (22)$$

where, Ω is the third order tensor of components

$$\Omega_{ilh}:=\ \Lambda_{ijL}(\mathrm{F}_{lL}\mathrm{B}_{jh}+\mathrm{B}_{jl}\mathrm{F}_{hL})$$

(as the last factor in the left-hand side is symmetric in the last two places).

Moreover, when an internal constraint like (21) is present, we follow classical theories and suppose that each quantity, which, in absence of the constraint, is ruled by a constitutive prescription (that is P, Y, Λ, k, ε, η, ψ) is now the sum of one *active* and one *reactive* component.

$$\mathrm{P} = \mathrm{P}_a + \mathrm{P}_r,\ \mathrm{Y} = \mathrm{Y}_a + \mathrm{Y}_r,\ \text{etc.} \tag{23}$$

and only the active component is bound through constitutive relations to the independent thermo-kinetic variables.

We wish to observe that, within the classic context, no constitutive equation can be proposed for an hyperelastic medium involving the second gradient of F (or B in the isotropic case) for the known incompatibility with the second law of thermodynamics, unless, for example, the interstitial working of Dunn and Serrin [26] is introduced in the balance equations.

Instead continua of second grade are acceptable if they are thought of as continua with latent microstructure: in fact, we assume here that, for the principle of equipresence, the overall response of the isotropic thermo-elastic porous material depends on the set $\mathcal{S} \equiv \{\mathrm{B}, \nabla\mathrm{B}, \theta, \nabla\theta\}$.

The additional request that the constraint is perfect, that is internally frictionless, is specified by the property that the contribution of the reactions to the inequality (13) is identically zero for every process allowed by the constraint (see Capriz [20]) that is:

$$\rho_*(\dot{\psi}_r + \eta_r\dot{\theta}) \leq$$
$$\mathrm{P}_r \cdot \dot{\mathrm{F}} + \mathrm{Y}_r \cdot \dot{\mathrm{B}} + \Lambda_r \cdot \nabla\dot{\mathrm{B}} - \frac{\mathrm{k}_r \cdot \nabla\theta}{\theta} \tag{24}$$

If we note that the constraint (21) leaves locally the choice of

$$\dot{\theta}, \mathrm{D}\left(= \mathrm{sym}(\dot{\mathrm{F}}\mathrm{F}^{-1})\right),$$

$$\frac{\partial^2\dot{\mathrm{x}}}{\partial\mathrm{x}^2}\left(= \left[\nabla(\dot{\mathrm{F}}\mathrm{F}^{-1})^T\right]\mathrm{F}^{-1}\right) \text{and } \nabla\theta$$

totally free, by using the equivalence $(22)_4$, we deduce the following relations:

$$\mathrm{sym}\,[\mathrm{P}_r\mathrm{F}^T + 2(\mathrm{Y}_r\mathrm{B} + \Lambda_r \odot \nabla\mathrm{B})] = 0$$
$$\rho_*\eta_r = 0,\ \rho_*\dot{\psi}_r = 0,\ \mathrm{k}_r = 0 \tag{25}$$

and, in components,

$$(\Lambda_r)_{ijL}\left(\mathrm{F}_{lL}\mathrm{B}_{jh} + \mathrm{B}_{jl}\mathrm{F}_{hL}\right) = 0 \tag{26}$$

Now, we have to check the compatibility of the constitutive prescriptions for the active components with the reduced Clausius-Duhem inequality (13); the same procedure used for the reactive components leads instead to

$$\psi_a = \hat{\psi}_a(\mathrm{B}, \nabla\mathrm{B}, \theta), \qquad \Lambda_a = \rho_*\frac{\partial\psi_a}{\partial\nabla\mathrm{B}},$$

$$\mathrm{sym}\,[\mathrm{P}_a\mathrm{F}^T + 2\,\mathrm{Y}_a\mathrm{B}] = 2\,\mathrm{sym}\left(\rho_*\frac{\partial\psi_a}{\partial\mathrm{B}}\mathrm{B}\right) \tag{27}$$

$$\eta_a = -\frac{\partial\psi_a}{\partial\theta},\ \mathrm{k}_a(\mathcal{S}) \cdot \nabla\theta \leq 0,$$

where the usual expression of the entropy η in terms of the free energy ψ ensues from Eqs. $(25)_{2,3}$ and $(27)_3$, while the residual Fourier inequality for the heating flux Eq. $(27)_5$ holds for every choice of the gradient of θ; we observe that the terms involving the second gradient of F in the constitutive equations are here not escluded, as within the classical context.

Now, in addition, we are able to specify the energy flux for our model of porous solids. We can use the micro-momentum balance (7) to eliminate the internal microactions Y from the equation for energy (9), then, by using the Legendre transform (2) and the constraint (21), we obtain the following equation:

$$\rho_*\dot{\tilde{\varepsilon}} = \mathrm{P} \cdot \dot{\mathrm{F}} + \rho_*\tilde{\lambda} - \mathrm{Div}\,\tilde{\mathrm{k}} \tag{28}$$

here we added the kinetic energy κ, due to the fluctuations of the pores' boundary, to the internal energy ε, the microstructural contribute $\mathrm{H} \cdot \dot{\mathrm{B}}$ to the radiant heating λ and the opposite of the so-called interstitial work flux vector $\mathrm{u} := \Lambda^T\dot{\mathrm{B}}$ (see Dunn et al. [26] and Giovine [27]) to the heat flux k, that is,

$$\tilde{\varepsilon} = \varepsilon + \kappa,\ \tilde{\lambda} = \lambda + \mathrm{H} \cdot \dot{\mathrm{B}},\ \tilde{\mathrm{k}} = \mathrm{k} - \mathrm{u} \tag{29}$$

In particular, when κ and H vanish, we recover the energy Eq. (1.14) of Dunn et al. [26] with the energy flux equal to the difference of interstitial work and the heat flux vectors, the stress power apart. We observe that the flux u expresses explicitly his dependence on the microstress tensor.

4. The Pure Field Equations

It is possible to use the balance Eqs. (7) and (10) and the relations (23) and (25) to eliminate the reactive parts of Eqs. (6) and (9) in order to arrive at a set of *pure*, that is reaction-free, consequences which alone are sufficient to study the evolution of our solid material with big pores. In fact, we have:

$$\begin{aligned} \mathrm{PF}^T &= \mathrm{sym}(\mathrm{P}_a\mathrm{F}^T) + \mathrm{sym}(\mathrm{P}_r\mathrm{F}^T) + \\ &+2\ \mathrm{skw}\ (\mathrm{BY} + \nabla\mathrm{B} \odot \Lambda) = \\ &= \mathrm{sym}[\mathrm{P}_a\mathrm{F}^T - 2(\mathrm{Y}_r\mathrm{B} + \Lambda_r \odot \nabla\mathrm{B})] - \\ &-2\ \mathrm{skw}\ (\mathrm{YB} + \Lambda_a \odot \nabla\mathrm{B} + \Lambda_r \odot \nabla\mathrm{B}\ ) = \\ &= \mathrm{sym}(\mathrm{P}_a\mathrm{F}^T + 2\mathrm{Y}_a\mathrm{B}) - 2\mathrm{YB} - \\ &-2\Lambda_r \odot \nabla\mathrm{B} - 2\ \mathrm{skw}\ (\Lambda_a \odot \nabla\mathrm{B}) \end{aligned} \tag{30}$$

and, by using the balance Eq. (7) in Eq. $(30)_3$, we obtain

$$\begin{aligned} \mathrm{PF}^T &= \mathrm{sym}(\mathrm{P}_a\mathrm{F}^T + 2\mathrm{Y}_a\mathrm{B}) + 2\mathrm{skw}\ (\nabla\mathrm{B} \odot \Lambda_a) + \\ &+2\rho_*\left[\frac{d}{d\tau}\left(\frac{\partial\chi}{\partial\dot{\mathrm{B}}}\right) - \frac{\partial\chi}{\partial\mathrm{B}} - \mathrm{H}\right]\mathrm{B} - \\ &-2(\mathrm{Div}\,\Lambda_a)\mathrm{B} - 2\ [\mathrm{Div}(\mathrm{B}\Lambda_r)]^T \end{aligned} \tag{31}$$

the last term of which, multiplied by F^{-T} has the divergence that vanishes for the relation (26) and the properties of symmetry of B and Λ_r .

Hence, with the use of constitutive relations (27), it is possible to introduce the reduced Piola stress tensor $\widehat{\mathrm{P}}$:

$$\begin{aligned} \widehat{\mathrm{P}} &:= \mathrm{P} + 2\ [\mathrm{F}^{-1}\mathrm{Div}(\mathrm{B}\Lambda_r)]^T = \\ &= 2\rho_*\left\{\left[\frac{d}{d\tau}\left(\frac{\partial\chi}{\partial\dot{\mathrm{B}}}\right) - \frac{\partial\chi}{\partial\mathrm{B}} - \mathrm{H}\right] - \mathrm{Div}\left(\frac{\partial\psi_a}{\partial\nabla\mathrm{B}}\right)\right\}\mathrm{F} + \\ &+2\rho_*\left[\mathrm{sym}\left(\frac{\partial\psi_a}{\partial\mathrm{B}}\mathrm{B}\right) + \mathrm{skw}\ \left(\nabla\mathrm{B} \odot \frac{\partial\psi_a}{\partial\nabla\mathrm{B}}\right)\right]\mathrm{F}^{-T} \end{aligned} \tag{32}$$

where the first two terms on the right hand side represent a tensor of micromomentum flux M due to a sort of surface flux of microinertia, the third one a tensor flux of external actions N and the remnants a partial stress tensor $\tilde{\mathrm{P}}$. Moreover, we observe that the expression for $\widehat{\mathrm{P}}\mathrm{F}^T$ is clearly not symmetric, in general.

The first pure field equation governing the processes for a thermo-elastic porous solid is the Cauchy's Eq. (6)

$$\rho_*\ddot{\mathrm{x}} = \rho_*\mathrm{f} + \mathrm{Div}\,\widehat{\mathrm{P}} \tag{33}$$

the second one being the balance equation for energy (9): in fact, by using the definition of free energy ψ, we have

$$\begin{aligned} &\rho_*\overline{(\psi + \theta\eta)}^{\cdot} = \\ &= \mathrm{P}\cdot\dot{\mathrm{F}} + \mathrm{Y}\cdot\dot{\mathrm{B}} + \Lambda\cdot\nabla\dot{\mathrm{B}} + \rho_*\lambda - \mathrm{Div}\ \mathrm{k} \end{aligned} \tag{34}$$

and then, from relations (25)-(27), we obtain the pure equation of evolution for the temperature:

$$\rho_*\theta\dot{\eta}_a = \rho_*\lambda - \mathrm{Div}\ \mathrm{k}_a \tag{35}$$

In Eqs. (33) and (35) there is no trace of effects due to the constraint since only the constitutive components of the fields appear.

5. A Constitutive Choice

To offer a simple example of application we consider a centro-symmetric isotropic thermo-elastic porous material with a density of kinetic co-energy χ and of active free energy ψ_a homogeneous and quadratic in $\dot{\mathrm{B}}$ and in the constitutive variables, respectively, of the following forms:

$$\chi := \frac{\chi_*}{4}\ \dot{\mathrm{B}}\cdot\ \dot{\mathrm{B}} \tag{36}$$

$$\begin{aligned} \psi_a &:= \frac{\lambda}{4}(\mathrm{tr}\ \mathrm{B})^2 + \frac{\mu}{2}\ \mathrm{tr}\ (\mathrm{B}^2) + \frac{1}{4}\Xi\cdot(\nabla\mathrm{B}\otimes\nabla\mathrm{B}) + \\ &+\frac{\beta}{4}\ (\theta - \theta_*)^2 + \frac{\gamma}{2}(\theta - \theta_*)\ \mathrm{tr}\ \mathrm{B} \end{aligned} \tag{37}$$

where θ_* is the referential value of the temperature; χ_*, λ, μ, β, γ are kinetic and thermo-elastic constants, respectively; Ξ an isotropic sixth-order tensor satisfying the following symmetries:

$$\Xi_{ijKlmN} = \Xi_{jiKlmN} = \Xi_{lmNijK} \tag{38}$$

(see Eq. (4.5) of Suhubi et al. [28] or Eq. (19) of Giovine [9]); ⊗ the tensor product so defined

$$(\Gamma\otimes\Sigma)_{ijKlmN} := \Gamma_{ijK}\Sigma_{lmN} \tag{39}$$

The heat flux vector k_a is chosen to satisfy a Fourier's type law, of constant ξ:

$$\mathrm{k}_a = -\xi\nabla\theta \tag{40}$$

With these hypotheses, the Legendre transform (2) for χ and the relations $(27)_{2,4}$ furnish the kinetic energy κ , the active part of the microstress tensor Λ_a and the symmetric part of the tensor $(\mathrm{P}_a\mathrm{F}^T + 2\ \mathrm{Y}_a\mathrm{B})$:

$$\kappa := \frac{\chi_*}{4}\,\dot{\mathrm{B}}\cdot\,\dot{\mathrm{B}}\,,\ \Lambda_a = \rho_*\Xi\nabla\mathrm{B} \qquad (41)$$

$$\mathrm{sym}(\mathrm{P}_a\mathrm{F}^T + 2\,\mathrm{Y}_a\mathrm{B}) =$$
$$= \rho_*[\lambda\,\mathrm{tr}\,\mathrm{B} + \gamma(\theta-\theta_*)]\mathrm{B} + 2\mu\mathrm{B}^2 \qquad (42)$$

where the Gibbs' relation $(27)_4$ and the Fourier's inequality $(27)_5$ give:

$$\eta_a = -\frac{1}{2}[\gamma\,\mathrm{tr}\,\mathrm{B} + \beta(\theta-\theta_*)]\ \text{ and } \xi \geq 0 \qquad (43)$$

Then, if the reference placement $\mathcal{B}_*$ of the material is homogeneous with the density ρ_* constant, the micromomentum flux *M*, the flux of external actions *N* and the partial stress tensor $\tilde{\mathrm{P}}$, defined after constitutive expression $(32)_2$, are given by:

$$\mathrm{M} = \rho_*\chi_*\ddot{\mathrm{B}}\mathrm{F},\ \ \mathrm{N} = -2\rho_*\mathrm{HF} \qquad (44)$$

$$\tilde{\mathrm{P}} = \rho_*\{[\lambda\,\mathrm{tr}\,\mathrm{B} + \gamma(\theta-\theta_*)]\mathrm{I} + 2\mu\mathrm{B} - \mathrm{Div}(\Xi\nabla\mathrm{B})\}\mathrm{F} +$$
$$+\rho_*[\mathrm{skw}\,(\nabla\mathrm{B}\odot\,\Xi\nabla\mathrm{B})]\mathrm{F}^{-T} \qquad (45)$$

where we can observe that MF^T, NF^T and $\tilde{\mathrm{P}}\mathrm{F}^T$ are not symmetric tensors, in general, and also that $\tilde{\mathrm{P}}\mathrm{F}^T$ depends on the first and the second gradient of the left Cauchy-Green tensor *B*, and so on those of *F*; therefore, for relation $(31)_2$, the reduced Piola stress tensor has the same properties.

At the end, by supposing the tensor Ξ in Eq. (45) as:

$$\Xi_{\mathrm{ijKlmN}} = \alpha\delta_{il}\delta_{jm}\delta_{KN} \qquad (46)$$

where α is a constant and δ is the delta di Kronecker, by inserting expressions (44) and (45) in Eq. $(32)_2$ and after in Eq. (33) and, further, by inserting relations (40) and $(43)_1$ in Eq. (35), Eqs. (33) and (35) reduce to the following pure field equations:

$$\ddot{\mathrm{x}} = \mathrm{f} + \mathrm{Div}\,\{[\lambda\,\mathrm{tr}\,\mathrm{B} + \gamma(\theta-\theta_*)]\mathrm{I} + 2\mu\mathrm{B} -$$
$$-\alpha\,\Delta\mathrm{B} + \chi_*\ddot{\mathrm{B}} - 2\,\mathrm{H}\}\mathrm{F}\ \ , \qquad (47)$$

$$\theta\big(\beta\dot{\theta} + \gamma\,\mathrm{tr}\,\dot{\mathrm{B}}\big) = 2\rho_*^{-1}\xi\,\Delta\theta - \lambda \qquad (48)$$

6. Conclusions

In this work we presented the general thermo-mechanical balance equations for a porous material with large voids viewed as a continuum with latent ellipsoidal microstructure. The tensorial order parameters, which describe the microstate, are completely constrained to the macrostrain and the microstructure disappears apparently. Hence some classical results are modified and some incompatibilities of the constitutive equations are left behind. Into details, the Cauchy stress tensor can be not symmetric, in general, and it can depend on higher derivatives of displacement and on acceleration gradients. Further, in the balance equation of momentum, as well as in that of energy, they appear fluxes of microstructural origin: in the first one a surface micromomentum flux and an external microforce flux, in the second one an interstitial work flux depending on the microstress.

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