Modified Iterative Method for Recovery of Sparse Multiple Measurement Problems

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Abstract: We consider the problem of constructing one sparse signal from a few measurements. This problem has been extensively addressed in the literature, providing many sub-optimal methods that assure convergence to a locally optimal solution under specific conditions. There are a few measurements associated with every signal, where the size of each measurement vector is less than the sparse signal's size. All of the sparse signals have the same unknown support. We generalize an existing algorithm for the recovery of one sparse signal from a single measurement to this problem and analyze its performances through simulations. We also compare the construction performance with other existing algorithms. Finally, the proposed method also shows advantages over the OMP (Orthogonal Matching Pursuit) algorithm in terms of the computational complexity.

Key words: Sparse signal recovery, iterative methods, multiple measurements.

1. Introduction

Sparse signal processing is an important problem which arises in many different problems such as sparse sampling, sparse coding, channel estimation and applications related to image processing. In these situations, the general problem can be posed as an under-determined sparse signal recovery problem. There are situations where the desired solution is not sparse but a sparse approximation of it exists as in a wide-band channel. There are other situations where the problem has an equivalent sparse representation in another domain. For instance, in the wavelet domain, an image can be represented with a sparse signal. All these problems fall into the category of a CS (compressed sensing) problem. This problem in its basic form consists of recovering an unknown sparse signal of interest from a measurement vector, referred to as the SMV (single measurement vector) problem. It has been shown that an exhaustive search to solve this problem is NP-hard [1]. A vast majority of literature has been dedicated to finding low-complexity

approximation algorithms whose solutions are close the optimal solution, as well. In the Gaussian noise structure, the l_1-norm minimization [2], LASSO [3] and greedy algorithms [4] are some examples. In the non-Gaussian noise structure, greedy algorithm [5] and Bayesian method [6] are some example approaches.

On the other hand, in some other applications, such as in array processing problems [7], neuromagnetic inverse problem that arises in MEG (magnetoencephalography) [8], linear inverse problem [8] and source location in sensor network [9] that the problem is more complex. In these problems, referred to as the MMV (Multiple Measurement Vector) model [10], the objective is to recover a sparse representation of signals from multiple measurement vectors. In this class, the signals are assumed to have a common sparsity profile. We assume that all of the sparse signals share the same support. Therefore, the problem is to find the sparse signals $x^i \in R^n$ given the measurement vectors $y^i \in R^m$ for $i = 1, 2, \ldots, N$, such that:

$$y^i = A^i x^i, \qquad i = 1, 2, \cdots, N$$

where $m < n$, and A^i are the sensing matrices.

The majority of the results for the MMV problem are

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obtained by generalizing the results of the SMV problems. For instance, it has been shown that the equivalence of l_0-norm and l_1-norm under some conditions as well as the uniqueness of the solution under both norms for the SMV problems extends to MMV problems, as well [10]. Likewise, recovery methods used for the SMV problems can be extended for the MMV problems, as well.

For instance, it has been shown that for particular problems, the OMP (orthogonal matching pursuit) algorithm can obtain the sparsest solution for the MMV problem [10]. The MP (matching pursuit) and the FOCUSS (FOCal underdetermined system solver) algorithms for the SMV problem have also been extended to MMV models [7, 10]. The convex relaxation methods and StOMP (Simultaneous Orthogonal Matching Pursuit) methods have been extended for MMV model in Ref. [11]. In another approach, the MMV prox algorithm has been proposed to solve MMV problems [12]. In this method, first the dual problem of the primal (2,1)-norm minimization problem is derived. Then, the dual optimization is reformulated as a minmax problem, and the problem is solved using the prox-method [13].

In this paper, we propose an algorithm for recovery of jointly sparse signals by applying a similar procedure as the extension of the OMP algorithm to solve the MMV problem for recovery of sparse signals with the same support. For this purpose, we use the StOMP algorithm as the base algorithm, and extend it to our problem of interest.

2. Method

In order to expedite recovery of sparse signals, greedy algorithms are introduced. For the SMV problem, instead of solving an ℓ_1 minimization problem through LASSO or linear programming, matching pursuit algorithm was introduced. Since the convergence condition of the MP is more restrict than other methods, it requires more measurements for exact recover. However, its computational complexity is superior. For instance, the number of iterations required for OMP is the same as the sparsity order, and each iteration just requires a pseudo-inverse, which has the complexity of $O(k^3)$ in each step. However, when both n and k are large, we are interested in algorithms with less computations.

OMP has been generalized to MMV problem using some modification in the support recovery step. In SMV, the index which has maximum correlation with the residual at each iteration is added to the support. In MMV, a summation is applied over all measurement vectors, and then the index having maximum correlation is selected [10]:

$$J = \text{argmax}\left(\sum_{i=1}^{N} |(A^i)^T r_k^i|\right),$$

$$T = T \cup J,$$

where r_k^i is the residual of the ith vector in the kth iteration, T is the common support, and we assumed that the columns of A^i have unit norm. Other steps are the same as the SMV recovery.

One way to decrease the complexity of OMP recovery method is to reduce the number of iterations when k is large. As mentioned earlier, OMP needs k iteration to reconstruct the original sparse signal completely. It is because of the fact that OMP recovers one support in each iteration. Therefore, we design a new procedure to add more than one index to the support in each iteration. The solution is to specify an adaptive threshold, and in every iteration, instead of choosing the index that gives the maximum correlation, we select all the indexes that their correlation is greater than that threshold [14]. This threshold can be a function of residual at each iteration. The StOMP procedure is summarized in Table 1.

The difference between StOMP and OMP is the threshold block. In fact, OMP picks the largest coefficient in each iteration and adds it to the support, but StOMP uses a threshold to find the appropriate support. In this situation, there is a chance that more than one component be added to the support in each

Table 1   The StOMP procedure.

1.	Set $i = 1$, $r = y$, $X = 0$, $t = 2.5$ and $T = \{\}$;
2.	while $i \leq 10$;
3.	$J = find(\|A^T r\| > t \frac{\|r\|_2}{\sqrt{m}})$;
4.	$T = T \cup J$;
5.	$X_T = (A_T^T A_T)^{-1} X_T\, r$;
6.	$r = y - AX$;
7.	end while

iteration. For the case that the number of non-zero elements, k, is large, StOMP needs much less iterations to recover the original signal; e.g. 10 iterations. Thus, instead of having overall complexity of $O(k^4)$ for OMP, the StOMP method has $O(k^3)$ complexity.

In order to generalize StOMP to the case of more than one sparse vector with common support, we use the same technique used for OMP. Based on that, at each iteration, instead of recovering support separately, we use all the measurement vectors together, which only changes step 3 in Table 1 as:

$$J = find\left(\sum_{i=1}^{N} |(A^i)^T r_k^i| > t\|r\|_2 \sqrt{\frac{N}{m}} \right)$$

Since there is a summation over N vectors, the threshold needs to be modified too, in which case a factor of $\sqrt{N}$ is added. Other steps are the same as before.

3. Simulation Result

In this section, we assess the performance of the proposed approaches for reconstruction of jointly sparse signals through simulations. For simulations, we set the size of the sparse signal to $n = 400$ and the number of measurements for each signal $m = 100$. For different number of non-zero elements and different number of sparse signals, we compare the probability of perfect reconstruction and simulation time. We use OMP [15], Cosamp [16], GHNM [5] and SP [17] as the benchmark and compare the performance of our method with these methods. In Fig. 1, we compare the performance of the method for a case that there is only one sparse signal. GHNM has the best performance in recovery, but its simulation time grows with the number of non-zero element. OMP has better performance than StOMP. For instance, when we have $k = 35$, both GHNM and StOMP have perfect recovery, but StOMP is faster than OMP. Thus, there is trade-off between complexity and recovery.

Fig. 2 compares these methods when there is more than one sparse vector. For this figure, the number of sparse signals is set to $N = 10$. It can be observed that having more sparse vectors with the same support improves the performance of all three methods. It is

(a)   (b)
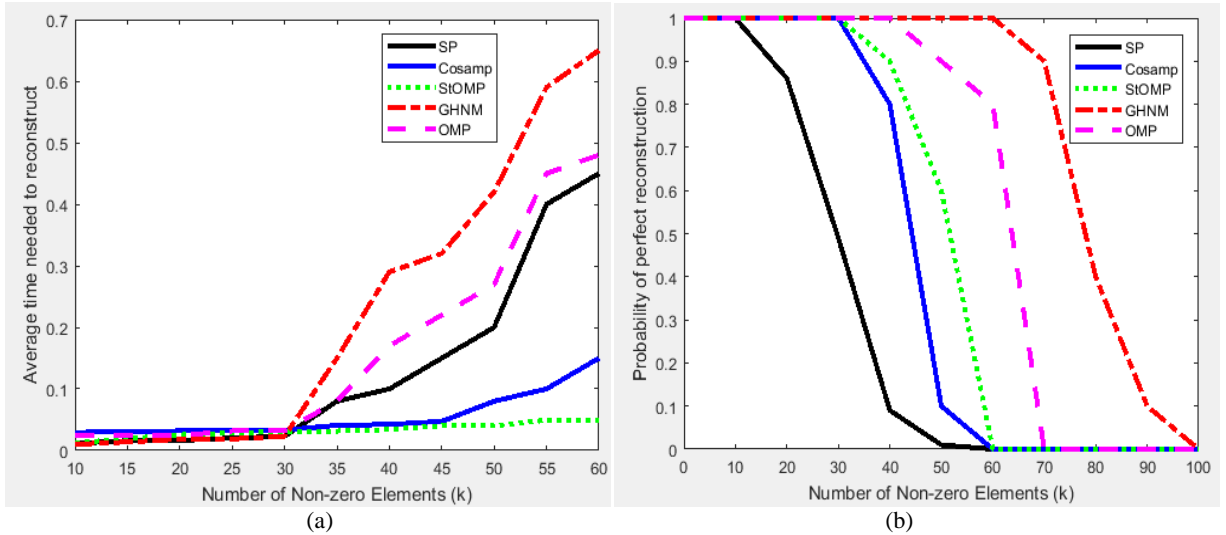
Fig. 1   Comparison of (a) percentage recovery and (b) simulation time of different methods in SMV against different number of non-zero elements and 10 sparse signals.
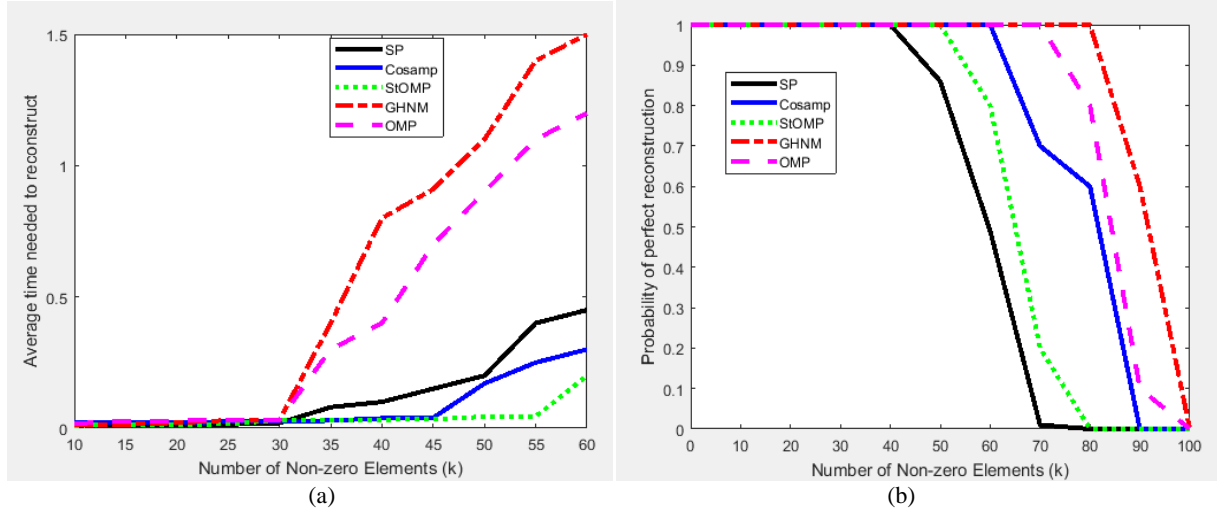
Fig. 2 Comparison of (a) percentage recovery and (b) simulation time of different methods in MMV against different number of non-zero elements and 10 sparse signals.
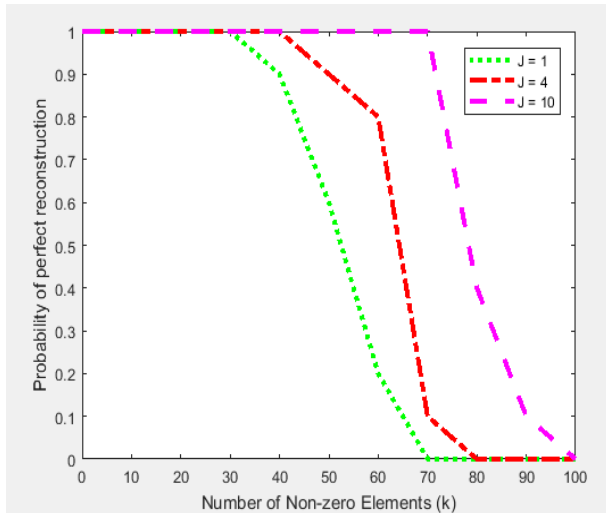
Fig. 3 Comparison of percentage recovery of StOMP in MMV against different number of non-zero elements and different number of sparse signals.

because of the fact that we used jointly recovery algorithms, and recovering every vector separately results in the same performance as the case of one sparse vector. Again, OMP compared to StOMP has better performance and needs more time. From this figure, we expect that increasing the number of sparse vectors improves the recovery performance.

To see the effect of increasing the number of sparse signals on the recovery performance, in Fig. 3, we compared the performance of these methods for 1, 4 and 10 signals. It is seen that increasing the number of sparse signals improves the performance greatly, and

we need less measurements in order to have the same recovery performance as the case of one vector.

4. Discussion

As mentioned earlier, recovery of jointly sparse signals is an NP-hard problem. A widely-used approach is to generalize sparse recovery algorithms to more than one sparse signal setting. This procedure has been done for some algorithms and we used the same technique to generalize another method. In the SMV setting, StOMP outperforms some of the existing methods in complexity, and since having more sparse signals increases complexity of the problem, we are interested in some algorithms with less complexity. Therefore, we generalized this method for jointly sparse recovery problems. These new methods have a simple structure and can be implemented easily.

5. Conclusion

In this paper, a new algorithm for recovery of jointly sparse signals was proposed. This method has been used previously for the recovery of one sparse signal, and we extended it to the new situation of multiple sparse signals with common support. StOMP outperforms OMP in terms of the computational complexity, but it requires more measurements to have the same recovery performance as OMP. Moreover, we

observed that more sparse vectors reduce the number of required measurements for perfect recovery.

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