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Inverse Problems in Solid Mechanics and Ultrasonic

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Abstract: The IP (inverse problem) in solid mechanics have emerged through the 20th century and measured. Nowadays has accept the classification of inverse problems in solid mechanics [1]: retrospective IP, boundary IP, coefficient IP, geometric IP. The IPs in theory of elasticity, theory of plasticity, theory of strengthening, fatigue theory, fracture theory micro-structural of polycrystalline materials are formulated and solved. The method of high frequency theory of ultrasonic waves is used. The results which are obtained here show, that all coefficients and geometrical characteristics of imperfections in the material are functions only of longitudinal and transversal velocities and attenuation of ultrasonic waves, which are measured, according ASTM E 494: 2015. The received relationships from this article could be used for NDE (non-destructive evaluation) of the coefficients and the geometrical characteristics of imperfections in polycrystalline materials.

Key words: IP, solid mechanics, ultrasonic.

1. Introduction

Classification of the IP (inverse problem) in solid mechanics includes [1].

(1) Retrospective IP. The functions $u_k = u_K(t_0); k = 1,2,3$ are evaluation;

(2) Coefficients IP. The function $a(x)$ is in the wave equation $\widetilde{p}_{tt} - \left(a(x)\widetilde{p}_x\right)_x = 0$, where $\widetilde{p}$—normalized sound pressure, is evaluation;

(3) Boundary IP. The stress vector p_i^n from $\sigma_{ij} n_j \big|_{S_X} = p_i^n; i = 1,2,3$ is evaluation;

(4) Geometric IP. The reflected area S_X in $\sigma_{ij} n_j \big|_{S_X} = p_i^n; i = 1,2,3$ is evaluation;

(5) Microstructure IP. The average value of grain size $\overline{D}$ in polycrystalline materials is evaluation. Retrospective IP and Boundary IP here are not view, because the conditions $u_k = u_K(t_0); k = 1,2,3$ and $\sigma_{ij} n_j \big|_{S_X} = p_i^n; i = 1,2,3$ are not considered because it is calibrations technics for ultrasonic examination technics [2].

In this article the IPs (2), (4) and (5) are formulated and solved as measured the acoustical characteristics of materials $(V_L; V_T; \alpha_L)$ [3].

2. Coefficients Inverse Problems

2.1 In Theory of Elasticity

The physical equations in theory of elasticity are $\sigma_{ij} = \lambda\ \delta_{ij}\ \varepsilon_{ii} + 2\ \ \mu\ \varepsilon_{ij}\ \ i, j = 1, 2, 3$. It is Hooke's law. The relationships between the physical modulus (λ, μ), the technical modulus (E, ν, G, K) and velocities of longitudinal and transversal ultrasonic waves (V_L, V_T) are [4]

$$\lambda + 2\mu = \rho V_L^2,\ \ \mu = \rho V_T^2 \tag{1}$$

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu},\ \nu = \frac{\lambda}{2(\lambda + \mu)};$$

$$G = \frac{\lambda(1 - 2\nu)}{2\nu},\ K = \lambda + \frac{2}{3}\mu. \tag{2}$$

Eqs. (1) and (2) are NDE of modulus of elasticity (λ, μ) and (E, ν, G, K) [4].

The velocity of transversal ultrasonic wave V_T is measured according to ASTM E 494 2015. In this case of the Snellius's law there is Eq. (3), in Fig. 1.

$$\frac{\left(V_T^X\right)^2}{\sqrt{V_1^2 - \left(V_T^X\right)^2 \sin^2 \vartheta_{1b}}} = \frac{W^X}{T(y).\sin\vartheta_{1b}}. \tag{3}$$

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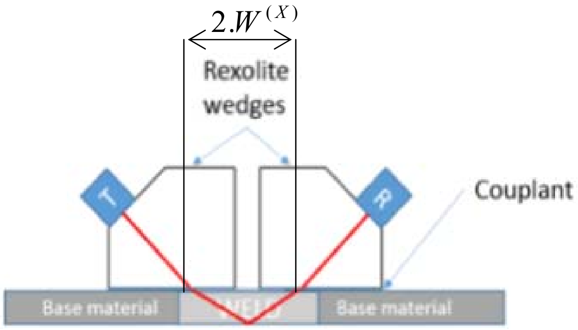

Fig. 1 Patent & US 5618994 A/1997.

where the value $W^{(X)}$ is distance between transducers (emitter-receiver), $t_T^{(X)}$ is time of transversal ultrasonic wave propagation. Eq. (3) introduced: ϑ_{1b} —angle in protector, V_1 —ultrasonic velocity in protector, $t_T^{(X)} - T(y) = 0$ is wave front, t_T —time of transversal ultrasonic wave propagation. After performing the necessary transformations of Eq. (3), the equation for V_T is obtained as Eq. (4).

$$\frac{(pV_T^{(X)})^2}{\sqrt{1-(pV_T^{(X)})^2}} - p.W^{(X)}/t_T = 0 \qquad (4)$$

where, p = $\dfrac{\sin\vartheta_{1b}}{V_1}$ —beam parameter.

The velocity V_L of propagation of longitudinal ultrasonic wave in material is obtained by Ref. [4].

$$V_L = \frac{2.\delta, mm}{\left(t_L^{(X)}\right), \mu s}, \qquad (5)$$

where, δ —thickness of tested object, $t_L^{(X)}$ —time of longitudinal ultrasonic wave propagation.

The coefficients inverse problem, in this case, is "evaluation of elastic modulus (E,ν,G,K) and (λ,μ) by measurement in Ref. [3] of longitudinal and transversal velocities of ultrasonic waves (V_L, V_T) propagation".

2.2 In Theory of Plasticity

In theory of plasticity the physical equations are $(\sigma_{ij} - \delta_{ij}\sigma_0)$ = $\Psi.(\varepsilon_{ij} - \delta_{ij}\varepsilon_0)$, , $i = 1.2.3$ (Henky's law), where Ψ is Henky's coefficient [5]. There are relationships $\Psi = \dfrac{1}{2G} + \varphi$, $\varphi = \dfrac{3}{2}.\dfrac{(\varepsilon_i)_p}{\sigma_i}$

$\rho\left\{V_L^{(E.t)}(\varepsilon_{xx})\right\}^2 = \Psi$. If $(\varepsilon_i;\sigma_i) \Rightarrow (\varepsilon_{xx};\sigma_{xx})$, then

$$\Psi = \frac{3}{2}.\left(\frac{1}{\sigma_S} - \frac{1-2.\nu}{3.E}\right) \qquad (6)$$

Eq. (6) is NDE of Henky's coefficient. In Ref. [5] the modified Holl-Petch formula (NDE of yield stress), for yield stress— σ_S , is obtained

$$\sigma_S = \sigma_0(\nu;G) + K_y(\nu;G).\left(\overline{D}\right)^{-1/2}, \qquad (7)$$

where for the technical modulus (E,ν,G,K) there are

$$\nu = \frac{0.5-(V_T/V_L)^2}{1-(V_T/V_L)^2}, G = \rho.(V_T)^2;$$

$$\frac{E}{\rho.V_T^2} = \left[\frac{3-4(V_T/V_L)^2}{1-(V_T/V_L)^2}\right].$$

For the non-destructive evaluation of average value of grain size— $\overline{D}$ by measuring the acoustical characteristics of materials (V_L, V_T, α_L) we have Popov's equation.

$$W_2(V_L,V_T)\, f^4\left(\overline{D}\right)^3 - \alpha_L = 0 \qquad (8)$$

where, $W_2(V_L,V_T) = \left[\dfrac{4.\pi^4}{1125}\left(\dfrac{V_T^4}{V_L^3}\right)\left(\dfrac{2}{V_L^5} + \dfrac{3}{V_T^5}\right)\right]$, i.e., $\overline{D} = \overline{D}\ (V_L, V_T, \alpha_L)$.

Eq. (8) is NDE of average value of grain size— $\overline{D}$ in polycrystalline materials.

The coefficient inverse problem, in this case, is: "evaluation of the Henky's coefficient Ψ by (V_L, V_T, α_L)".

2.3 In Theory of Strengthening

If the strengthening model is $\sigma_{xx} = b.\varepsilon_{xx}^n$, then parameters (b,n) are obtained

$$\boldsymbol{b} = \frac{\varphi(\nu).HB}{\left(\frac{0.2}{100}+\frac{1-2.\nu}{3.E}\sigma_S\right)^{1/5}}$$

$$\left\{(1-n)\left(\frac{n}{1-n}\right)^n\right\} = \frac{1}{3}HB^{0.989} \qquad (9)$$

where

$$\varphi(\nu)=\left\{\frac{1}{2}(1-2.\nu)+\frac{2}{9}(1+\nu).[2.(1+\nu)]^{1/2}\right\} \quad [4],$$

$\sigma_B = 0.34HB^{0.989}$ [4].

Eq. (9) is NDE of parameters (b,n).

The modified Stroh's relationship is

$$\sigma_B = \sigma_0^{(B)}(V_L,V_T) + K_y^{(B)}(V_L,V_T).\left(\overline{D}\right)^{-1/2}$$

$$\sigma_B = 0.34HB^{0.989} \qquad (10)$$

where $\sigma_0^{(B)}(V_L,V_T)$ and $K_y^{(B)}(V_L,V_T)$.

The coefficients inverse problem, in this case, is "The parameters (b,n) are obtained by measured $(V_L,V_T,HB,\overline{D})$".

2.4 In Theory of Fatigue

The basic result in theory of fatigue is Wöhler's curve .

$$\sigma_{\max} = \sigma_{-1} + a.N_1^{-b} \qquad (11)$$

where σ_{-1} is fatigue limit, $(\sigma_{\max};N_1)$ are mechanical stress and number of cycles to destruction, $(a;b)$—parameters. The Eq. (11) in the form of Vagapov is

$$N_1(\sigma_{\max}-\sigma_{-1})^2 = \frac{E^2}{16}\ln^2\left(\frac{1}{1-\psi}\right) \qquad (12)$$

where for fatigue limit σ_{-1} .

$$\sigma_{-1} = \sigma_0^{(-1)}(V_L,V_T)+K_y^{(-1)}(V_L,V_T).\left(\overline{D}\right)^{-1/2} \qquad (13)$$

The coefficients $\sigma_0^{(-1)}(V_L,V_T)$ and $K_y^{(-1)}(V_L,V_T)$ in Ref. [4] are obtained as function of (V_L,V_T). From Eq. (8) therefore $\overline{D}=\overline{D}(V_L,V_T,\alpha_L)$. Eq. (13) is NDE of fatigue limit σ_{-1} .

For Young's modulus and relative contraction—(E,ψ) are obtained the relationships

$$\frac{E}{\rho.V_T^2}=\left[\frac{3-4(V_T/V_L)^2}{1-(V_T/V_L)^2}\right], \text{ ASTM E494. 2015}$$

$$A\ \psi^n - B\ \psi - C \approx 0,\ n=4/15 \text{ [6];} \qquad (14)$$

$$A=\left(\frac{0.2}{100}+\frac{\sigma_S(V_L;V_T)}{E(V_L;V_T)}\right)^{-4/15},$$

$$B=\frac{7}{5}.\left(\frac{\sigma_B(V_L;V_T)}{\sigma_S(V_L;V_T)}\right),\ C=\left(\frac{\sigma_B(V_L;V_T)}{\sigma_S(V_L;V_T)}\right).$$

Eqs. (13) and (14) are NDE of parameters (σ_{-1},E,ψ) in Vagapov's curve (12).

The coefficients inverse problem, in this case, is "evaluation of the functions $\sigma_0^{(-1)}(V_L;V_T)$ and $K_y^{(-1)}(V_L;V_T)$, by measured $(V_{:L};V_T;\alpha_L)$, where (α_L) —attenuation coefficients at longitudinal ultrasonic wave propagation".

2.5 In Fracture Theory

In Ref. [7] for the value of stress intensive factor—K_{IC} is obtained

$$K_{IC}^2 = \chi(\overline{D}.)\tau_S.\frac{E}{1-\nu^2}.\ln\left(\frac{1}{1-\psi}\right) \qquad (15)$$

where, $\chi(\overline{D})$ is function of average value of grain size $\overline{D}$, $\tau_S=\frac{1}{\sqrt{3}}\sigma_S$ —yield stress, $(E;\nu)$—Young's modulus and Poison's coefficient, ψ —relative contraction. After conducting the necessary transformations Eq. (15) is down to

$$K_{IC}^{*}\left(\overline{D}\right) \approx \sigma_0^{K_{IC}}\left(V_L;V_T\right)+K_y^{K_{IC}}\left(V_L;V_T\right).\left(\overline{D}\right)^{-1/2} \quad (16)$$

where, $\sigma_0^{K_{IC}}\left(V_L;V_T\right) = \left(\frac{\sigma_0}{\sigma_S^{REF}}\right)..U\left(V_L,V_T\right)$;

$K_y^{K_{IC}}\left(V_L;V_T\right)=\left(\frac{K_y}{\sigma_S^{REF}}\right)..U\left(V_L;V_T\right)$;

$$U\left(V_L;V_T\right) = \left(\frac{E\left(V_L;V_T\right)}{E^{REF}}\right) . \left(\frac{1-\nu^2\left(V_L;V_T\right)}{1-\left(\nu^{REF}\right)^2}\right)^{-1} .$$

$\left(\frac{\psi\left(V_L;V_T\right)}{\psi^{REF}}\right)$; $K_{IC}^{*} = \left(\frac{K_{IC}}{K_{IC}^{REF}}\right)^2$

Eq. (16) is NDE of stress intensive factor—K_{IC} by $K_{IC}^{*}\left(\overline{D}\right)$.

For low carbon steels, the reference values $(\ )^{REF}$ are approximately $K_{IC}^{REF} \approx 54MPa.mm^{1/2}$ $\sigma_0^{(REF)} \approx 72MPa$; $K_y^{(REF)} \approx 23.9MPa.mm^{1/2}$ and $\left(E^{REF}/\nu^{REF}/\psi^{REF}\right) \approx$ $\left(\sim 201.MPa/\sim 0.256/\sim(65-75\%)\right)$. Therefore we have $K_{IC}^2 = K_{IC}^2\left(V_L;V_T;\alpha_L\right)$.

3. Geometrical Inverse Problems

3.1 Evaluation of Reflected Area by Acoustical Tract Analysis

For the calculated the reflected area—S_d, Eq. (17) is used.

$$\left(\frac{D_J S_{IIII}\cos\gamma/\cos\beta}{\lambda_T^2(r+r_{IIII})^2}\right).S_d-(\tilde{p})=0 \quad (17)$$

where, D_J —omission coefficient by energy, S_{IIII} —area of piezo-plate, (γ,β) —incident and orientation angles respectively, λ_T —wave length, $(r;r_{IIII})$—ultrasonic roads to the reflector and through the protector of the transducer respectively, $(\tilde{p}_S)$—scattering acoustic field, in Fig. 2.

The Geometrical Inverse Problem, in this case, is "calculation of the reflected area S_d at given D_J, λ, S_{IIII}, $(\gamma,\beta),(r_{IIII})$ and measured $(\tilde{p}_S;r)$".

3.2 Evaluation of Depth of Crack

The cracks are boundary for tested sample in Fig. 3.

The ultrasonic diffraction method for calculating the depth of crack—*h* Eq. (18) is used.

$$h = \left\{\varphi_{Up}\left(p.V_T\right)\right\}.\left(W^{(X)}\right)^2 \quad (18)$$

where, $\varphi_{U_P}\left(p.V_T^{(X)}\right)\frac{1}{2}.\left(\frac{p.V_T^{(X)}}{\varphi\left(pV_T^{(X)}\right)}\right)^2-1$, $\varphi\left(pV_T\right)-p.W^{(X)}/t_p=0$ [4].

The function—$\varphi(pV_T)$ and the beam parameter—*p* are described in Eq. (4). The geometrical inverse problem, in this case, is "to calculate the depth of cracks *h* by beam parameter—*p* and measure of $\left(V^T,W^{(X)},t_T\right)$".

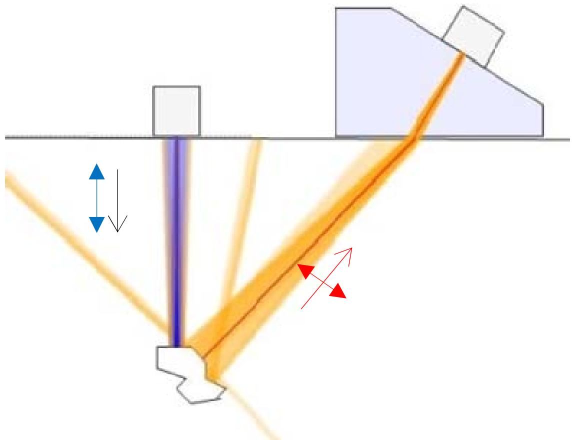

Fig. 2 Ultrasonic reflected method.

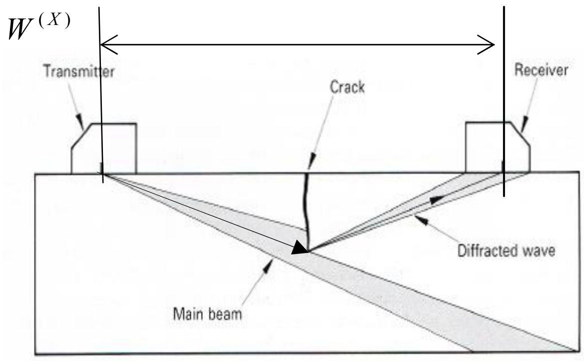

Fig. 3 Ultrasonic diffraction method.

4. Microstructure Inverse Problem

The basic characteristics of polycrystalline microstructures [8] are average value of grain size— $\overline{D}$. The value $\overline{D}$ is random value with Weibull's density of distribution i.e., $p(D,a,b) = \frac{b}{a}\left(\frac{D}{a}\right)^{b-1}\exp\left[-\left(\frac{D}{a}\right)^{b}\right]$. If $p(D,a,b)$ is known, then $\overline{D}$ is defined

$$\overline{D}\,(D,a,b) = \int_{0}^{\max D} D.p(D,a,b)dD \ . \quad (19)$$

The IP, in this case, is "evaluated as the density of distribution $p(D,a,b)$ by means measured of (V_L, V_T, α_L) and evaluation of $\overline{D}$ and solve of Fredholm's integral equations, 1-st kind (14) for $p(D,a,b)$, through $a(\overline{D})$ and $b(\overline{D})$ [4]".

5. Conclusions

In this paper, it was obtained equations for evaluation of the velocities of longitudinal and transversal ultrasonic wave propagation.

The relationships between following mechanical properties: modulus of elasticity, Henky's coefficient, parameters in strengthening theory, Wöhler's curve, fatigue limit, coefficient in fracture mechanics and acoustics characteristics of materials are obtained for NDE.

The geometrical inverse problem for reflected area and crack depth are solved by means reflected and diffraction methods respectively.

The micro-structural IP by means mathematical-ultrasonic dualism is solved.

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