

Research Synergy of Multi-Stage Mathematical and Informational Tasks on Fractal Geometry*

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The paper emphasized the importance of students' creativity development based on the principle of variability and synergetic approach. The basic characteristics of the variability principle are also presented. The contribution of Russian teachers to studying the concepts of "mental flexibility" and the "principle of variability" is noted. This paper presents the ways of students' mental flexibility development when elaborating the algorithms of constructing the Julia sets with numerous applications in various areas of the real world in different environments. The Julia sets of quadratic polynomials are considered first and the algorithms of constructing the Julia sets are proposed. During studying Chebyshev polynomials and elaborating their Julia sets, the possibility of developing students' mental flexibility is substantiated on the specific problems. Attention to the filling Julia sets is given. The algorithms of constructing the Julia sets of rational functions are considered and the examples are given. A description of the algorithms is given with the aid of computer program. The parameter values, at which the Julia sets become smooth for quadratic polynomials, are specified. It is established that the segments of a real straight line are the Julia sets of Chebyshev polynomials. The examples of the Julia sets with a fractal structure are given. The relationship of the Julia sets with random maps is indicated. Construction of the Julia sets provides an opportunity to develop students' mental flexibility—one of the most important creative qualities of personality.

Keywords: creativity, Julia set, chaotic displaying, mental flexibility, multi-stage mathematical and informational task, attractor

Introduction

The first multi-stage mathematical tasks were considered by Klyaklya (2003). Later, Sekovanov (2006) continued studies of Klyaklya and introduced the concept of "multi-stage mathematics and informational task." The analysis and research of which turned out to be sensitive to development of creativity and creative activity of students in the attainment of mathematical knowledge and procedures (Sekovanov, 2006; Smirnov, Soloviev, & Burakova, 2002).

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During development of which the use of both mathematical techniques and information communication technologies (ICT), multi-stage mathematical and informational tasks is provided as the following (Sekovanov, 2006):

1. Representing specially prepared sequence of tasks, exercises, problems, and didactic situations that connect different types of creative mathematical activity;
2. Carrying out of mathematical and computer experiments;
3. Performance of laboratory work on mathematics;
4. Decision of non-standard mathematical problems;
5. Predicting the results of mathematical activity;
6. Online searching with each other.

We understand multi-stage mathematical and informational tasks as a laboratory within which the creative mathematical and informational activity aimed at development of creative qualities of bachelor, master, and post-graduate student. This approach to studying of Julia sets' polynomials and rational functions, closely related to fractal geometry and chaos theory. It also to being the key components of rapidly developing research area—synergistic studying the laws of self-organization of complex dynamical systems. According to our opinion, the stated approach to studying Julia sets, closely related to fractal sets, would be in demand by students and post-graduate students, since the appendices of fractals are currently observed in all spheres—from education to modern technologies. When performing the multi-stage mathematical and information tasks, a student's world is forming, the intelligence, convergent, and divergent thinking is developing, and the skill to preview the results of mathematical activity is generating.

Technologies and Procedures

Accomplishment of mathematical and informational task is provided during three stages. The scheme-plan of this task is shown in Figure 1. It is noted that while performing a multi-stage mathematical and informational task, the development of algorithms of Julia sets elaboration using high-level programming languages is provided as well as mathematical packages, which will positively influence on the development of mental flexibility—the most important creative quality of personality.

We shall describe each of these steps and indicate the solution of didactic tasks aimed at development of creativity of students. A brief description of fractals on the complex plane is given at the first stage.

Mathematical Construction of Julia Sets

Julia sets were described at the beginning of the last century. However, it was not until 50 years later that the above-mentioned sets were constructed via a computer. Moreover, the configuration of the above-mentioned sets amazed the entire mathematical world. It should be noted that Julia sets are constructed with the aid of computer programs and they are one of the most beautiful mathematical objects which has beneficial effect on the development of creativity and competence of students and post-graduate students (Sekovanov, 2006, 2012, 2015, 2016; Smirnov, Soloviev, & Burakova, 2002; Sekovanov, Salov, & Samokhov, 2011; Sekovanov & Ivkov, 2013; Sekovanov & Fomin, 2016).

Let the function $f(z) = z^2 + c$ be given. The Julia set for $f(z)$ indicating as $J(f)$ is determined as $J(f) = \partial \{z: f^{(n)}(z) \rightarrow \infty, n \rightarrow \infty\}$, where ∂ —the border of infinity attraction area and $f^{(n)}(z) = f(f^{(n-1)}(z))$, $n = 1, 2, \dots$

Filling Julia sets, consisting of those points on the complex plane, which orbits are caught, are often considered. The border of the filling Julia set will be the Julia set.

Computer Design of Julia Sets

Examples of filling Julia sets are presented in Figures 2 and 3.

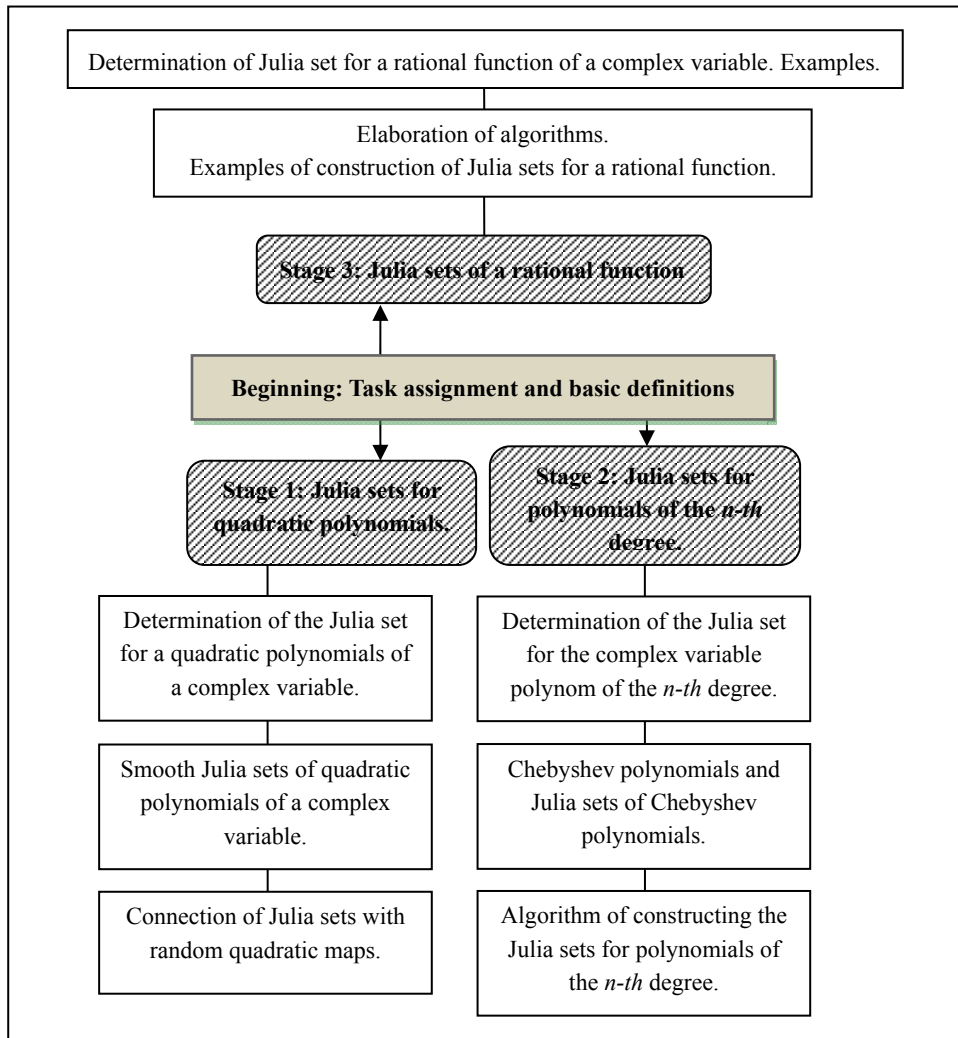


Figure 1. Schematic plan of multi-stage mathematical and informational task “Julia sets of polynomials and rational functions.”

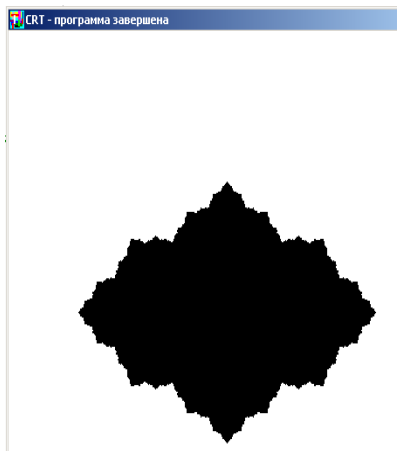


Figure 2. Julia set for function $f(z) = z^2 - 0.5$.

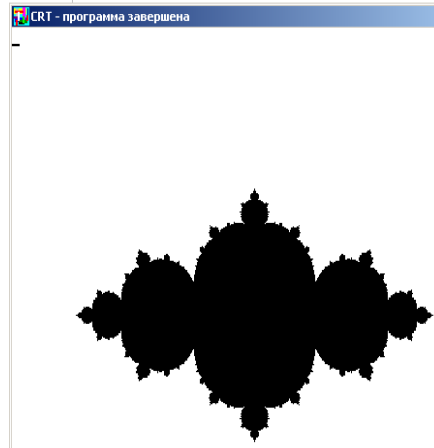


Figure 3. Julia set for function $f(z) = z^2 - 0.7$.

It is important to emphasize that the Julia set (the boundary of the filling Julia set) has a fractal structure. However, students should know that there are smooth Julia sets as well.

For example, the Julia set is a circle $|z| = 1$ a smooth set for the function $f(z) = z^2$.

Here is the proof scheme.

We have $f^2(z) = (z^2)^2 = z^4$ and let $|z| = 1$. Since $|z|^4 = |z^2| |z^2| = 1$, then $|f^2(z)| = 1$. Consequently, $f^2(z)$ is on the unit circle centered at the origin. Similarly, we can check that the points $f^3(z), f^4(z), \dots$ are also located on the unit circle. Note that $f^n(z) = z^{2^n}$.

It is easy to verify that the sequence $f^n(z) = f(f^{n-1}(z))$ tends to ∞ if $|z| > 1$. When $|z| < 1$, this sequence will tend to 0. In the case considered, the unit circle centered on the origin will be the Julia set, since $\partial\{z : f^n(z) \rightarrow \infty, n \rightarrow \infty\} = |z| = 1$ and a circle of unit radius centered on the origin will be a filling Julia set.

Further, it is helpful to ensure for the students of function $f(z) = z^2$ randomness on a unit circle centred on the origin and using the teaching aids (Sekovanov, Salov, & Samokhov, 2011; Sekovanov & Ivkov, 2013). Also, it is helpful to establish for the students that the segment $[-2; 2]$, on which the function $g(z)$ is chaotic, is the Julia set for the function $g(z) = z^2 - 2$.

The definition of the Julia set for a polynomial of the n -th degree is given and the examples of filling Julia sets Figures 4 and 5 are presented at the second stage.

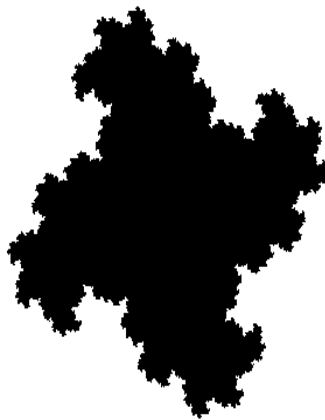


Figure 4. Julia set for function $f(z) = z^4 - 0.1194 + 0.6289i$.



Figure 5. Julia set for function $f(z) = z^5 - 0.1194 + 0.6289i$.

Furthermore, the students are invited to consider the Chebyshev polynomials in general terms. Let $P_1(z) = z$, $P_2(z) = z^2 - 2$, $P_3(z) = z^3 - 3z$, $P_4(z) = z^4 - 4z^2 + 2$, we will inductively define Chebyshev polynomials by the formula $P_{n+1}(z) + P_{n-1}(z) = z P_n(z)$.

We note that $P_3(z) = z(z^2 - 2) - z = z^3 - 3z$, $P_4(z) = z(z^3 - 3z) - (z^2 - 2) = z^4 - 4z^2 + 2$, $P_5(z) = z(z^4 - 4z^2 + 2) - (z^3 - 3z) = z^5 - 5z^3 + 5z$.

It is further proposed to prove that the segment $[-2; 2]$ is the Julia set of the Chebyshev polynomial.

Computer Design of Chebyshev Polynomial

The proof schemes are as the following:

1. We shall note first that $z = h(w) = w + \frac{1}{w}$ displays the unit circle S of the radius 1 centered on the origin ($|w| = 1$) per the segment $[-2; 2]$;
2. We shall note that if $|w| > 1$, then $w + \frac{1}{w} \in \mathbb{C} \setminus [-2; 2]$ and the circle exterior bounded via a circumference of a unit circle centered on the origin is displayed on the segment exterior $[-2; 2]$;
3. It is verified that $P_n(m)(h(w)) = w^{n^m} + \frac{1}{w^{n^m}}$ and if $h(w) \notin [-2; 2]$, then $\lim_{n \rightarrow \infty} (P_n(m)(h(w))) = \lim_{n \rightarrow \infty} \left(w^{n^m} + \frac{1}{w^{n^m}} \right) = \infty$;
4. $P_n([-2; 2]) \subset [-2; 2]$ is being set.

By virtue of above proof schemes, it is drawn a conclusion that the segment $[-2; 2]$ is the Julia set for the polynomial P_n .

Note that the Julia set for P_n is a not a fractal and segment $[-2; 2]$ will also be a filling Julia set of a polynomial P_n .

A computer program of the Julia set construction in Pascal ABC environment is considered at the final stage of the second phase.

```

uses crt, Graph ABC;
var height, width: integer;
var i, j, k: integer;
var dx, dy, x_max, x_min, y_max, y_min, kef, scall: real;
function f(x) (z: complex): complex; begin f(x) = z^2 - 0.7; end;
var c: complex;
var iter: integer;
begin
width = 500; height = 500; setwindowsize (width and height); iter = 100; scall = 2;
if (width >= height) then

```

```

begin kef: = width/height; x_min: = scall * -kef; x_max: = scall * kef; y_min: = scall * -1; y_max: = scall;
end else if (height > width) then
begin
kef := height/width; x_min: = scall * -1; x_max: = scall; y_min: = scall * -kef;
y_max: = scall * kef;
end;
dx: = (x_max - x_min)/width; dy: = (y_max - y_min)/height;
for i: = 1 to width do
for j: = 1 to height do begin
c: = (x_min + i * dx ,y_min + j * dy);
try
for k: = 1 to iter do c: = fx (c); kef: = power (c.re, 2) + power (c.im, 2);
if (kef < 4) then
setpixel (i, j, clblack );
except
//setpixel(i, j, white);
end;
end;
end.

```

We shall note that this program is a universal one to the extent that we will obtain the Julia set corresponding to it via changing polynomials in it.

The definition of the Julia set for a rational function as a closure of the repelling periodic points is given at the third stage. Attention of the students is paid to the fact that this definition coincides with the one given for the polynomials above. The examples of Julia sets for some rational functions, being the straight lines, are given (see Figures 6 and 7).

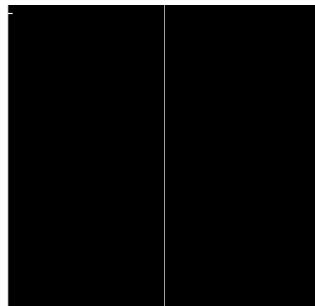


Figure 6. Julia set for $f(z) = z - \frac{z^2 - 1}{2z}$.



Figure 7. Julia set for $f(z) = z - \frac{z^2 + 1}{2z}$.

Results

The examples of Julia sets of rational functions with a fractal structure are given in Figures 8 and 9.

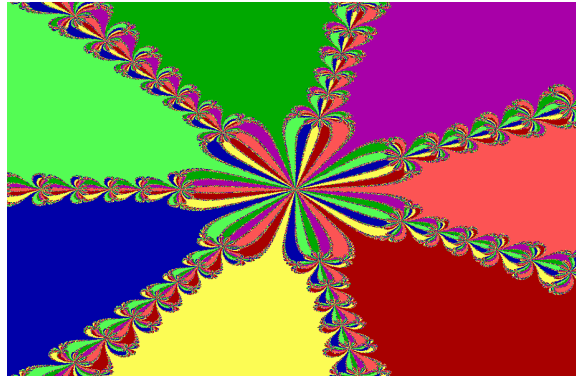


Figure 8. Julia sets of function $f(z) = z - \frac{z^7 - 1}{7z^6}$.

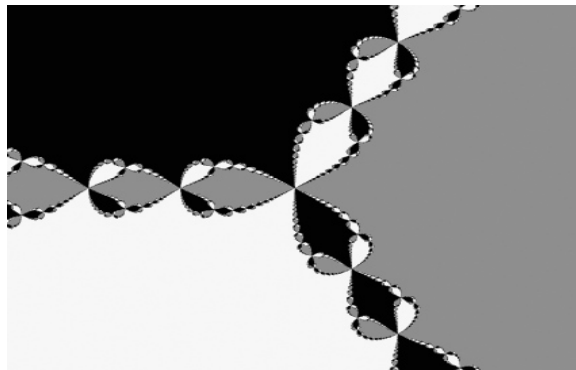


Figure 9. Julia sets of function $f(z) = z - \frac{z^3 - 1}{3z^2}$.

It is easy to verify that the points $1, -\frac{1}{2} + i \frac{\sqrt{3}}{2}, -\frac{1}{2} - i \frac{\sqrt{3}}{2}$ are the attracting fixed points of the function $f_2(z)$.

Let us designate via $z_i, i = 0, 2, \dots, 6$ the roots of the seventh degree of unity.

The Julia sets of rational functions defined as the boundaries of the domains of attraction for attracting fixed points are presented in Figures 8 and 9. So, we can find that $J(f_0) = \partial(z_0) = \partial(z_2) = \dots = \partial(z_6)$ in Figure 8, while $J(f_2) = \partial(1) = \partial(-\frac{1}{2} + i \frac{\sqrt{3}}{2}) = \partial(-\frac{1}{2} - i \frac{\sqrt{3}}{2})$ in Figure 9. In each of the two cases, the Julia set of the rational function is the border of an attraction area of attracting fixed points (attractors).

Let us consider the rational function further $R_q(x) = \left(\frac{x^2 + q - z}{2x + q - z}\right)^2$.

Let us indicate the algorithm of constructing the Julia sets for the function R_q . The construction of this algorithm consists of the following stages:

1. Each number z of the complex plane is applied to the corresponding function;
2. The obtained value is applied to the same function at the next iteration again;
3. The distance to the attractor is estimated after each iteration based on the obtained result of substitution and the conclusion on the trajectory is drawn;

4. Each point of the plane under study is designated via corresponding color depending on the convergence to a particular attractor.

The Julia sets of rational functions $R_4(z)$ and $R_3(z)$ are presented in Figures 10 and 11 (Sekovanov, 2012).

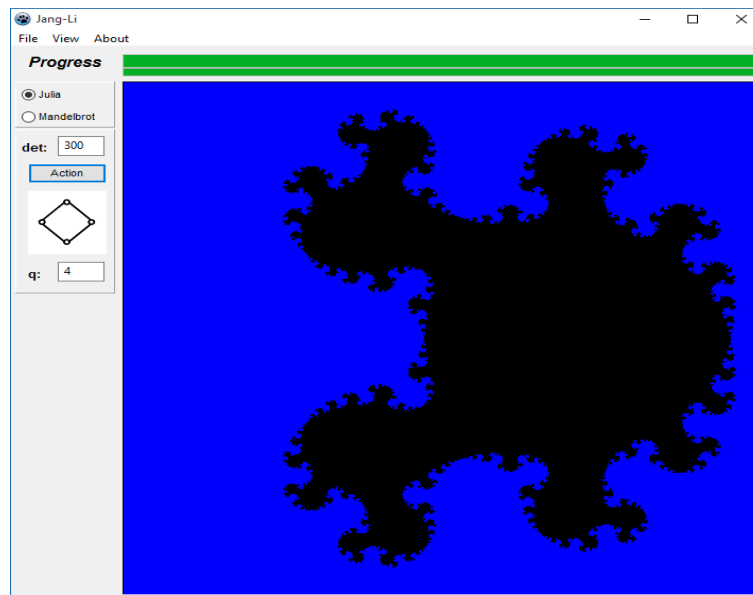


Figure 10. Julia set for $R_4(z)$.

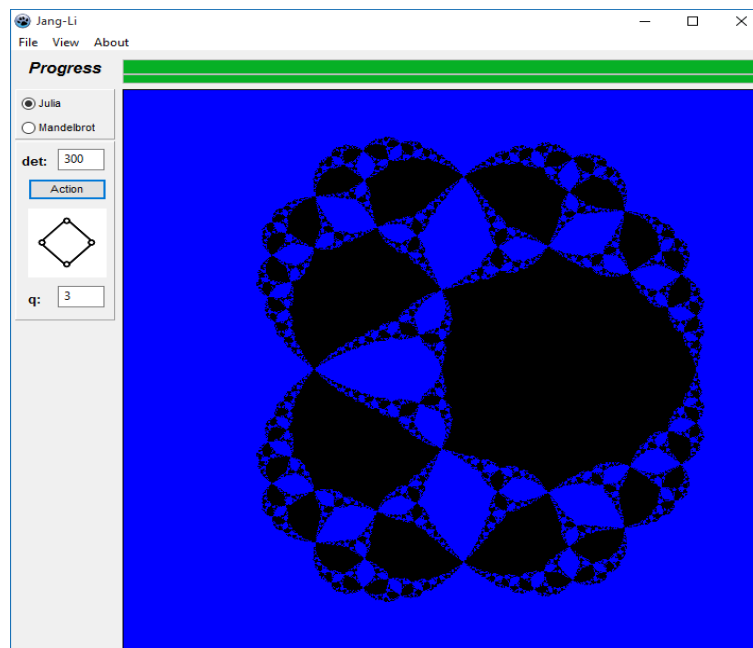


Figure 11. Julia set for $R_3(z)$.

The above-mentioned Julia sets are constructed with using of programming languages. It is useful to consider the algorithms of generating the Julia sets with the help of mathematical packages in order to develop mental flexibility. The Julia sets for the quadratic function $f(z) = z^2 + W$ in the package are presented in Figures 12 and 13.

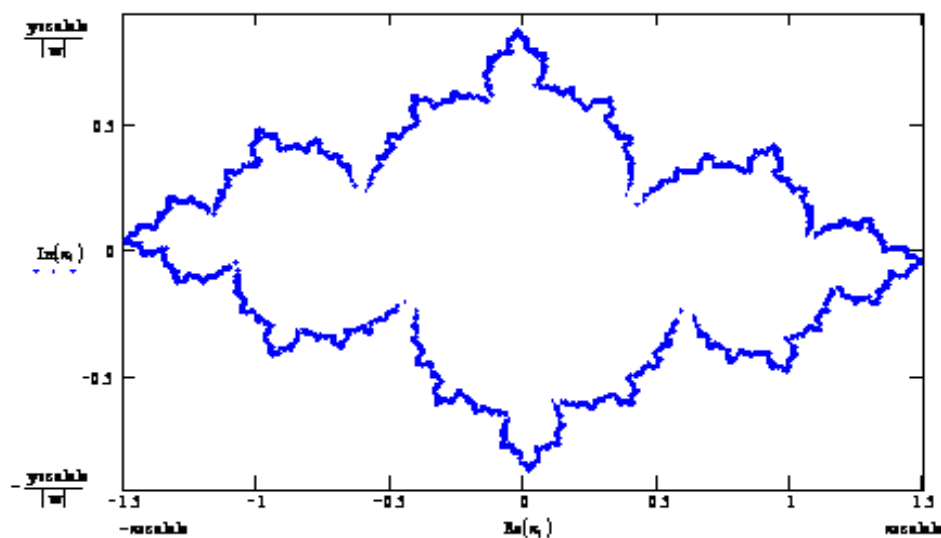


Figure 12. Julia set for the function $W = -0.738 + 0.082i$.

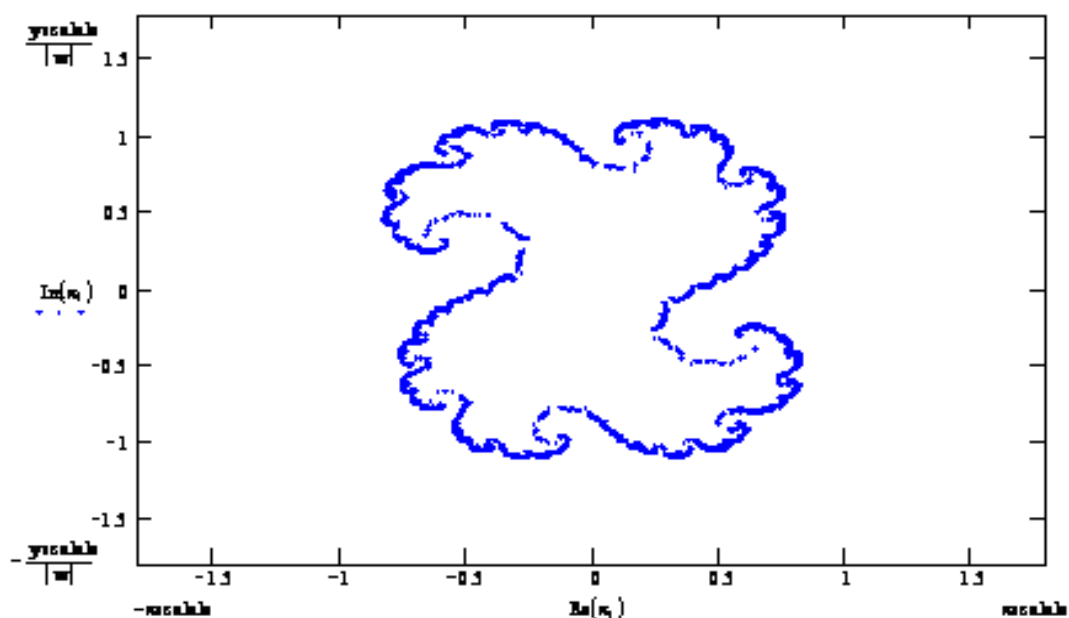


Figure 13. Julia set for the function $W = 0.33 + 0.62i$.

Conclusion

It should be noted that the implementation of a multi-stage mathematical and informational task “Julia sets of polynomials and rational functions” is aimed at developing students’ creativity and increasing their motivation to both mathematics and informatics due to the fact that the study of Julia sets is only possible when using both mathematical methods and ICT.

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