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# A Note on Special Functions Related to Lotka / Negative Volterra Equations

Takashi Fujiwara

**Abstract:** We study special functions related to Lotka-Volterra equations and negative Volterra equation intro-duced from zero curvature representations . At first we show the relationships between Lotka-Volterra equations introduced from zero curvature representations and symmetric orthogonal polynomials. Sec-ondarily, we describe the relationships between negative Volterra equations with a special solutions and cylinder functions.

**keyword.** Lotka-Volterra equations, Negative Volterra equations, symmetric orthogonal polynomials, cylinder functions, three term recurrences

## 1 Introduction

In this paper we study Lotka-Volterra equations and negative Volterra equations related to special functions $\{\phi_n(\lambda, t)\}$ satisfying the identities

$$\lambda\phi_n(\lambda, t) = \phi_{n+1}(\lambda, t) + u_n(t)\phi_{n-1}(\lambda, t),$$

where we denote $n$ as a natural number and $\lambda$, $t$ as a real number. At first we consider the relationships between Lotka-Volterra equations

$$\frac{d}{dt}u_n(t) = u_n(t)(u_{n+1}(t) - u_{n-1}(t))$$

and symmetric orthogonal polynomials satisfying three term recurrence

$$\lambda P_n(\lambda, t) = P_{n+1}(\lambda, t) + u_n(t)P_{n-1}(\lambda, t), \qquad P_{-1}(\lambda, t) = 0, \quad P_0(\lambda, t) = 1$$

For this relationships there are many investigations([1], [3], [6], [8]). Here we describe this relationships using zero curvature representations([3], [6]).

Secondarily, we consider the relationships between negative Volterra equations

$$\frac{d}{dt}u_n(t) = q_{n-1}(t) - q_n(t), \qquad u_n(t) = q_n(t)q_{n-1}(t)$$

and special functions satisfying the identities

$$\lambda\psi_n(\lambda, t) = \psi_{n+1}(\lambda, t) + u_n(t)\psi_{n-1}(\lambda, t)$$

Negative Volterra equations were introduced by Pritula and Vekslerchik[4]. Here giving special solutions $q_n(t) = \dfrac{t}{2n}$ for negative Volterra equations, we show that there is relationships between negative Volterra equations and *cylinder functions*([2],[9]).

Within this framework, we introduce in the next section the notion of symmetric orthgonal polynomials and describe the relationships between Lotka-Volterra equations and symmetric orthogonal polynomials. In section 3, we derive a relationships between *cylinder functions* and negative Volterra equations with special soulutions $q_n(t) = \dfrac{t}{2n}$.

Osaka Municipal Higashi Yodo technical highschool

# 2 Orthogonal polynomials

In this section we refer the general property of orthogonal polynomials([1],[4]). At first we define the moments, and we will construct orthogonal polynomials by using the determinants with elements as we denote the moments.

## 2.1 Moment and Hankel determinant for orthogonal polynomials

We let the vector space $V = \{x^n \in \mathbf{R} \mid n = 1, 2, 3, \cdots\}$. We define the linear functional $\mathcal{L} : V \longrightarrow \mathbf{R}$. By using this linear functional $\mathcal{L}$ , we define **moments** $s_k$ as

$$s_k = \mathcal{L}(x^k) = \int_{x \in \Lambda} x^k d\mu(x) \qquad (k = 0, 1, 2, 3, \cdots),$$

where we denote $\mu$ as Borel measure on $\Lambda = \mathrm{supp}\mu \subset \mathbf{R}$. Here we want to define **Hankel determinant** with the elements as we denote this moment $\{s_k\}$.

$$\tau_n = \begin{vmatrix} s_0 & s_1 & \cdots & s_{n-1} \\ s_1 & s_2 & \cdots & s_n \\ \vdots & \vdots & \cdots & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-2} \end{vmatrix}$$

## 2.2 Monic orthogonal polynomials represented by determinants

We denote $\{P_n(x)\}$ as a monic $n$ th polynomials with respect to variant $x$. Now we can get the monic orthogonal polynomials $P_n(x)$ using the Hankel determinants $\tau_n$

$$P_n(x) = \frac{1}{\tau_n} \begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}$$

It follows from it that we have the **orthogonality**

$$\mathcal{L}(x^m P_n(x)) = \frac{\tau_{n+1}}{\tau_n} \delta_{nm} \quad (m = 0, 1, 2, \cdots n)$$

## 2.3 Three term recurrences for orthogonal polynomials

Assume that we have the equality

$$x P_n(x) = a_n^{(n+1)} P_{n+1}(x) + a_n^{(n)} P_n(x) + a_n^{(n-1)} P_{n-1}(x) + \cdots + a_n^{(1)} P_1(x) + a_n^{(0)} P_0(x)$$

We operate $\mathcal{L}[\ \cdot \ P_{n-1}(x)]$ on both side of above equality and then it follows from the orthogonality $\mathcal{L}(x^m P_n(x)) = \dfrac{\tau_{n+1}}{\tau_n} \delta_{nm}$ that we have

$$\mathcal{L}(x P_n(x) P_{n-1}(x)) = a_n^{(n-1)} \mathcal{L}(P_{n-1}(x) P_{n-1}(x))$$

Thus, we obtain

$$\begin{aligned} \mathcal{L}(x^n P_n(x)) &= a_n^{(n-1)} \mathcal{L}(x^{n-1} P_{n-1}(x)) \\ \frac{\tau_{n+1}}{\tau_n} &= a_n^{(n-1)} \frac{\tau_n}{\tau_{n-1}} \\ a_n^{(n-1)} &= \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2} \end{aligned}$$

Moreover we operate $\mathcal{L}[\ \cdot \ P_n(x)]$, and then we can get

$$\mathcal{L}(x P_n(x) P_n(x)) = \mathcal{L}(a_n^{(n)} P_n(x) P_n(x))$$

By Laplace expansion of Hankel determinant for orthogonal polynomials and the orthogonality we have the equality

$$\mathcal{L}\left[x\left(x^n-\frac{1}{\tau_n}\begin{vmatrix} s_0 & s_1 & \cdots & s_{n-2} & s_n \\ s_1 & s_2 & \cdots & s_{n-1} & s_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-3} & s_{2n-1} \end{vmatrix} x^{n-1}\right)\frac{1}{\tau_n}\begin{vmatrix} s_0 & s_1 & \cdots & s_{n-1} & s_n \\ s_1 & s_2 & \cdots & s_n & s_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-2} & s_{2n-1} \\ 1 & x & \cdots & x^{n-1} & x^n \end{vmatrix}\right] = a_n^{(n)}\frac{\tau_{n+1}}{\tau_n}$$

Moreover we obtain

$$\frac{1}{\tau_n}\begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-1} \\ s_{n+1} & s_{n+2} & \cdots & s_{2n+1} \end{vmatrix} - \frac{\tau_{n+1}}{\tau_n^2}\begin{vmatrix} s_0 & s_1 & \cdots & s_{n-1} \\ s_1 & s_2 & \cdots & s_n \\ \vdots & \vdots & \cdots & \vdots \\ s_{n-2} & s_{n-1} & \cdots & s_{2n-3} \\ s_n & s_{n+1} & \cdots & s_{2n-1} \end{vmatrix} = a_n^{(n)}\frac{\tau_{n+1}}{\tau_n}$$

Thus we have

$$a_n^{(n)} = \frac{1}{\tau_{n+1}}\begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-1} \\ s_{n+1} & s_{n+2} & \cdots & s_{2n+1} \end{vmatrix} - \frac{1}{\tau_n}\begin{vmatrix} s_0 & s_1 & \cdots & s_{n-1} \\ s_1 & s_2 & \cdots & s_n \\ \vdots & \vdots & \cdots & \vdots \\ s_{n-2} & s_{n-1} & \cdots & s_{2n-3} \\ s_n & s_{n+1} & \cdots & s_{2n-1} \end{vmatrix}$$

In the case of $k = 0, 1, 2, \cdots n-2$, we operate $\mathcal{L}[\quad \cdot \quad P_n(x)]$, and then we have

$$\begin{aligned} \mathcal{L}(xP_n(x)P_k(x)) &= \mathcal{L}(a_n^{(k)}P_k(x)P_k(x)) \\ \mathcal{L}(x^{k+1}P_n(x)) &= a_n^{(k)}\frac{\tau_{k+1}}{\tau_k} \qquad k = 0, 1, 2, \cdots n-2 \\ a_n^{(k)} &= 0 \qquad (k = 0, 1, 2, \cdots n-2) \end{aligned}$$

For all monic orthogonal polynomials, we have three term recurrences

$$xP_n(x) = P_{n+1}(x) + a_n^{(n)}P_n(x) + a_n^{(n-1)}P_{n-1}(x)$$

## 2.4 Favard's theorem

At first we put $a_n^{(n)} = b_n$ and $a_n^{(n-1)} = u_n$. Then we have the equalities

$$xP_n(x) = P_n(x) + b_nP_n(x) + u_nP_{n-1}(x)$$

Next we put $P_{-1}(x) = 0$ and $P_0(x) = 1$.
Assume $b_n$ is real and $u_n > 0$ for all $n = 1, 2, \cdots$.
Then the zeros of the polynomials generated by $xP_n(x) = P_{n+1}(x) + b_nP_n(x) + u_nP_{n-1}(x)$, $P_{-1}(x) = 0$, $P_0(x) = 1$ are real and simple. Furthermore the zeros of $P_n$ and $P_{n-1}$ interplace.
Let

$$x_{N,1} > x_{N,2} > \cdots > x_{N,N}$$

be the zero of $P_N(x)$.
**Theorem (Favard)** *If we have the three term recurrence*

$$xP_n(x) = P_{n+1}(x) + b_nP_n(x) + u_nP_{n-1}(x)$$

*with $P_{-1}(x) = 0$ and $P_0(x) = 1$, and we assume*

$$b_n \in \mathbf{R} \quad \text{and} \quad u_n > 0$$

*for all $n > 0$, then there exists a distribution function $\mu$ such that we have*

$$\int_{x\in\Lambda} P_n(x)P_m(x)d\mu(x) = \zeta_n\delta_{nm}$$

*where* $\zeta_n$ *denotes*

$$\zeta_n = \prod_{j=1}^{n} u_j$$

To prove this theorem we introduce the following lemma

**Lemma 1** (**Helly's selection theorem**[5]) *We assume the sequence* $\{x_{N,j}\}_{j=1}^{n} \subset \Lambda$ *Let* $\Theta$ *be a family of functions defined on* $\Lambda$ *and satisfying the conditions*

$$\sup_{\Delta} \sum_{k=1}^{n} \mid f(x_k) - f(x_{k-1}) \mid \leq C \qquad \sup_{x \in \Lambda} \mid f(x) \mid \leq M \quad (f \in \Theta)$$

*for suitable* $C$ *and* $M$*, where the least upper bounded is taken over all finite partitions*

$$\Delta : \quad x_{N,n} < x_{N,n-1} < x_{N,n-2} < \cdots < x_{N,1}$$

*Then* $\Theta$ *contains a sequence which converges for ever* $x \in \Lambda$.

*proof* Let $f = h - g$, where $h$ is the total variation of $f$ on $[x_{N,n}, x]$. Then the functions $h$ corresponding to all $f \in \Theta$ satisfy the condition of the lemma, since

$$\sup_{\Delta} \sum_{k=1}^{n-1} \mid h(x_{N,k+1}) - h(x_{N,k}) \mid = \sup_{\Delta} \sum_{k=1}^{n-1} \mid f(x_{N,k+1}) - f(x_{N,k}) \mid \leq C \qquad \sup_{x \in \Lambda} \mid h(x) \mid \leq C$$

Now we choose a sequence $\{f_n\}$ from $\Theta$ such that $h_n$ converges to a limit $h$ on $\Lambda$. Then the functions $g_n = h_n - f_n$ are also satisfy the conditions of the lemma. Therefore $\{f_n\}$ contains a subsequence $\{f_{n_k}\}$ such that $\{g_{n_k}\}$ converges to a limit $g^*$ on $\Lambda$. Then $\lim_{k \to \infty} f_{n_k}(x) = f^*(x)$, where $f^* = h^* - g^*$.

Let $r_1, \cdots, r_n, \cdots$ be the rational points on $\Lambda$. It follows from the conditions of the lemma that the set of numbers

$$f(r_1) \qquad (f \in \Theta)$$

is bounded. Hence there is a sequence of functions $\{f_n^{(1)}\}$ converging at the point $r_1$. Similarly, $\{f_n^{(1)}\}$ contains a subsequence $\{f_n^{(2)}\}$ converging at the point $r_2$ as well as at $r_1$, $\{f_n^{(2)}\}$ contains a subsequence $\{f_n^{(3)}\}$ converging at the point $r_3$ as well as at $r_1$ and $r_2$, and so on. Then the *diagonal sequence* $\{F_n\} = \{f_n^{(n)}\}$ will converge at every rational point of $\Lambda$. The limit of this sequence is a function $F$, definded only at the points $r_1, \cdots, r_n, \cdots$. We complete the definition of $F$ at the remaining points of $\Lambda$ by setting

$$F(x) = \lim_{r \to x-0} F(r),$$

if $x$ is irrational where we denote $r$ as rational number.

The resulting function $F$ is then the limit of $\{F_n\}$ at every continuity point of $F$. In fact, let $x^*$ be such point. Then given any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\mid F(x^*) - F(x) \mid < \frac{\varepsilon}{6} \tag{1}$$

if $\mid x^* - x \mid < \delta$. Let $r$ and $r'$ be rational numbers such that

$$x^* - \delta < r' < x^* < r'' < x^* + \delta$$

and let $n$ be so large that

$$\mid F_n(r') - F(r') \mid < \frac{\varepsilon}{6}, \qquad \mid F_n(r'') - F(r'') \mid < \frac{\varepsilon}{6} \tag{2}$$

It follows from (1) and (2) that we have

$$\begin{aligned} \mid F_n(r') - F_n(r'') \mid &\leq \mid F_n(r') - F(r') \mid + \mid F(r') - F(x^*) \mid + \mid F(x^*) - F(r'') \mid + \mid F(r'') - F_n(r'') \mid \\ &< \frac{2}{3}\varepsilon \end{aligned}$$

Moreover we have

$$\begin{aligned} | F(x^*) - F_n(x^*) | &\leq | F(x^*) - F(r^{'}) | + | F(r^{'}) - F_n(r^{'}) | + | F_n(r^{'}) - F_n(x^*) | \\ &< \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{2}{3}\varepsilon \\ &= \varepsilon \end{aligned}$$

Therefore

$$\lim_{n\to\infty} F_n(x^*) = F(x^*),$$

since $\varepsilon > 0$ is arbitrary.

Thus we have constructed a sequence $\{F_n\}$ of functions in $\Theta$ converging to a limit function $F$ everywhere except possibly at discontinuity points of $F$. This lemma was proved.

Now we introduce a sequence of right continuous step functions$\{\nu_n\}$ by

$$\nu_N(\infty) = 0, \quad \nu_N(x_{N,j}+0) - \nu_N(x_{N,j}-0) = \rho(x_{N,j})$$

where we define a sequence

$$\rho(x_{N,j}) = \frac{\zeta_{N-1}}{P_N^{'}(x_{N,j})P_{N-1}(x_{N,j})} \qquad j = 1, 2, \cdots N$$

**Lemma 2** The moments $\displaystyle\int_R x^j d\nu_N(x)$, $1 \leq j \leq 2N-2$ do not depend on $\zeta_k$ for $k > \lfloor (j+1)/2 \rfloor$, where $\lfloor a \rfloor$ denotes the integer part of $a$.

*proof* The fixed $j$ choose $N > 1 + j/2$ and write $x^j$ as $x^s x^\ell$, with $0 \leq \ell, s \leq N-1$. Then express $x^s$ and $x^\ell$ as linear combinations of $P_0(x)$, $\cdots$, $P_{N-1}(x)$. Thus the evaluation of $\displaystyle\int_R x^j d\nu_N(x)$ involves only $\zeta_0$, $\cdots$, $\zeta_{\lfloor (j+1)/2 \rfloor}$

(**Proof of Favard's theorem**)
Since

$$1 = \zeta_0 = \int_R d\nu_N(x) = \nu_N(\infty) - \nu_N(-\infty)$$

then the $\nu_N$'s are uniformly bounded. From Helly's selection theorem it follows that there is subsequence $\eta_{N_k}$ which converges to a distribution function $\mu$. It follows from **Lemma 2** and Helly's selection theorem that $\{\{\nu_N\} - \{\eta_{N_k}\}\}$ also converges to $\mu$. It is clear that the limiting function $\mu$ of any subsequence will have infinitely many points of increase.

Thus Favard's theorem was proved.

# 3 Zero curvature representations

There are many investigations of the relationships between integrable systems and special functions. For example, there is the relationships between Toda molecule equations and orthogonal polynomials. At first, we have three term recurrences for orthogonal polynomials

$$P_{n+1}^t(\lambda) + b_n(t)P_n^t(\lambda) + u_n(t)P_{n-1}^t(\lambda) = \lambda P_n^t(\lambda)$$

Here we put

$$\Phi_n^t(\lambda) = \begin{pmatrix} P_n^t(\lambda) \\ -P_{n-1}^t(\lambda) \end{pmatrix},$$

then we obtain the linear equations

$$\Phi_{n+1}^t(\lambda) = L_n^t(\lambda)\Phi_n^t(\lambda),$$

where we denote

$$L_n^t(\lambda) = \begin{pmatrix} \lambda - b_n(t) & u_n(t) \\ -1 & 0 \end{pmatrix}.$$

Next we assume that we have

$$\frac{d}{dt}\Phi_n^t(\lambda) = A_n^t(\lambda)\Phi_n(\lambda),$$

where we denote

$$A_n^t(\lambda) = \begin{pmatrix} \alpha_n^t(\lambda) & \beta_n^t(\lambda) \\ \gamma_n^t(\lambda) & \delta_n^t(\lambda) \end{pmatrix}.$$

Then we differentiate both hand of $\Phi_{n+1}^t(\lambda) = L_n^t(\lambda)\Phi_n^t(\lambda)$, then we can get the equations

$$\frac{\partial}{\partial t} L_n^t(\lambda) - A_{n+1}^t(\lambda)L_n^t(\lambda) + L_n^t(\lambda)A_n^t(\lambda) = 0$$

Therefore we have four equations

$$\begin{cases} \dot{b}_n(t) = (\lambda - b_n(t))(\alpha_n^t(\lambda) - \alpha_{n-1}^t(\lambda)) + \beta_{n+1}^t(\lambda) + u_n(t)\gamma_n^t(\lambda) \\ \dot{u}_n(t) = u_n(t)(\alpha_{n+1}^t(\lambda) - \delta_n^t(\lambda)) - (\lambda - b_n^t)\beta_n^t(\lambda) \\ \delta_{n+1}^t(\lambda) = \alpha_n^t(\lambda) + (\lambda - b_n(t))\gamma_{n+1}^t(\lambda) \\ \beta_n^t(\lambda) = -\gamma_{n+1}^t(\lambda)u_n(t) \end{cases}$$

It follows from four equations that we have two equations

$$\begin{cases} \dot{b}_n(t) = (\lambda - b_n(t))(\alpha_n^t(\lambda) - \alpha_{n+1}^t(\lambda)) - \gamma_{n+2}^t(\lambda)u_{n+1}(t) + u_n(t)\gamma_n^t(\lambda) \\ \dot{u}_n(t) = u_n(t)(\alpha_{n+1}^t(\lambda) - \alpha_{n-1}^t(\lambda) - (\gamma_n^t(\lambda) - \gamma_{n+1}^t(\lambda))\lambda \\ \qquad +\gamma_n^t(\lambda)b_{n-1}(t) - b_n(t)\gamma_{n+1}^t(\lambda)). \end{cases} \tag{3}$$

Moreover we put $\alpha_n^t(\lambda) = 0$, $\gamma_n^t(\lambda) = 1$, we can get Toda lattice equations

$$\begin{cases} \dot{b}_n(t) = u_n(t) - u_{n+1}(t) \\ \dot{u}_n(t) = u_n(t)(b_{n-1}(t) - b_n(t)) \end{cases}$$

Here we put the moments

$$s_k(t) = \mathcal{L}^t(\lambda^k) = \int_{x\in\Lambda} x^k \frac{\exp(-V(x,t))}{\displaystyle\int_{y\in\Lambda} \exp(-V(y,t))d\mu(y)} d\mu(x),$$

where we denote $V(x,t)$ as a function on $(x,t) \in \Lambda \times \mathbf{R}$. Then we can obtain the solution of Toda equations represented by Hankel determinants

$$\begin{aligned} b_n(t) &= \frac{1}{\tau_{n+1}(t)} \begin{vmatrix} s_0(t) & s_1(t) & \cdots & s_n(t) \\ s_1(t) & s_2(t) & \cdots & s_{n+1}(t) \\ \vdots & \vdots & \cdots & \vdots \\ s_{n-1}(t) & s_n(t) & \cdots & s_{2n-1}(t) \\ s_{n+1}(t) & s_{n+2}(t) & \cdots & s_{2n+1}(t) \end{vmatrix} \\ &- \frac{1}{\tau_n(t)} \begin{vmatrix} s_0(t) & s_1(t) & \cdots & s_{n-1}(t) \\ s_1(t) & s_2(t) & \cdots & s_n(t) \\ \vdots & \vdots & \cdots & \vdots \\ s_{n-2}(t) & s_{n-1}(t) & \cdots & s_{2n-3}(t) \\ s_n(t) & s_{n+1}(t) & \cdots & s_{2n-1}(t) \end{vmatrix} \end{aligned} \tag{5}$$

and

$$u_n(t) = \frac{\tau_{n+1}(t)\tau_{n-1}(t)}{\tau_n(t)^2}$$

where we denote

$$\tau_n(t) = \begin{vmatrix} s_0(t) & s_1(t) & \cdots & s_{n-1}(t) \\ s_1(t) & s_2(t) & \cdots & s_n(t) \\ \vdots & \vdots & \cdots & \vdots \\ s_{n-1}(t) & s_n(t) & \cdots & s_{2n-2}(t) \end{vmatrix}$$

Following Aptekarev, Branquinho and Marcellan [1], $V(x,t) = xt$ holds.

Next we will introduce Lotka-Volterra equations. At first, when we put $b_n(t) = 0$ in the equations(1), we assume that $u_n(t)$ become $v_n(t)$. Then we obtain two equations

$$\begin{cases} 0 = \lambda(\alpha_n^t(\lambda) - \alpha_{n+1}^t(\lambda)) - \gamma_{n+2}^t(\lambda)v_{n+1}(t) + v_n(t)\gamma_n^t(\lambda) \\ \dot{v}_n(t) = v_n(t)(\alpha_{n+1}^t(\lambda) - \alpha_{n-1}^t(\lambda) - (\gamma_n^t(\lambda) - \gamma_{n+1}^t(\lambda))\lambda). \end{cases} \quad (6)$$

Next we put $\alpha_n^t(\lambda) = v_n(t)$ and $\gamma_n^t(\lambda) = -\lambda$, then we can obtain

$$\frac{d}{dt}v_n(t) = v_n(t)(v_{n+1}(t) - v_{n-1}(t)).$$

Now we refer relationships between moments $s_n(t)$and $v_n(t)$. We assume that we have

$$s_{2k-1}(t) = \mathcal{L}^t\left[\lambda^{2k-1}\right] = \int_{x\in\Lambda} x^{2k-1}\frac{\exp(-V(x,t))}{\displaystyle\int_{y\in\Lambda}\exp(-V(y,t))d\mu(y)}d\mu(x) = 0 \qquad (k = 1, 2, 3\cdots)$$

Then we get $b_n(t) = 0$. Moreover we have the equality

$$\tau_n(t) = f_n(t)f_{n-1}(t)$$

where we denote

$$\tau_{2n}(t) = \begin{vmatrix} s_0(t) & 0 & s_2(t) & \cdots & s_{2n-2}(t) & 0 \\ 0 & s_2(t) & 0 & \cdots & 0 & s_{2n}(t) \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ s_{2n-2}(t) & 0 & s_{2n}(t) & \cdots & s_{4n-4}(t) & 0 \\ 0 & s_{2n}(t) & 0 & \cdots & 0 & s_{4n-2}(t) \end{vmatrix},$$

$$\tau_{2n-1}(t) = \begin{vmatrix} s_0(t) & 0 & s_2(t) & \cdots & 0 & s_{2n-2}(t) \\ 0 & s_2(t) & 0 & \cdots & s_{2n-2}(t) & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & s_{2n-2}(t) & 0 & \cdots & s_{2n-6}(t) & 0 \\ s_{2n-2}(t) & 0 & s_{2n}(t) & \cdots & 0 & s_{4n-4}(t) \end{vmatrix}$$

and

$$f_{2n-1}(t) = \begin{vmatrix} s_0(t) & s_2(t) & \cdots & s_{2n-2}(t) \\ s_2(t) & s_4(t) & \cdots & s_{2n}(t) \\ \vdots & \vdots & \cdots & \vdots \\ s_{2n-2}(t) & s_{2n}(t) & \cdots & s_{4n-4}(t) \end{vmatrix}$$

$$f_{2n}(t) = \begin{vmatrix} s_2(t) & s_4(t) & \cdots & s_{2n}(t) \\ s_4(t) & s_6(t) & \cdots & s_{2n+2}(t) \\ \vdots & \vdots & \cdots & \vdots \\ s_{2n}(t) & s_{2n+2}(t) & \cdots & s_{4n-2}(t) \end{vmatrix}$$

Therefore we obtain

$$v_n(t) = \frac{\tau_{n+1}(t)\tau_{n-1}(t)}{\tau_n(t)^2} = \frac{f_{n+1}(t)f_n(t)f_{n-1}(t)f_{n-2}(t)}{f_n(t)f_{n-1}(t)f_n(t)f_{n-1}(t)} = \frac{f_{n+1}(t)f_{n-2}(t)}{f_n(t)f_{n-1}(t)}$$

Following Aptekarev, Branquinho and Marcellan [1], $V(x,t) = x^2t$ holds.

## 3.1 Symmetric orthogonal polynomials

It follows from $\dot{\Phi}_n^t(\lambda) = A_n^t(\lambda)\Phi_n^t(\lambda)$ that we have

$$\begin{aligned} \begin{pmatrix} \dot{\varphi}_n \\ -\dot{\varphi}_{n-1} \end{pmatrix} &= \begin{pmatrix} v_n & \lambda v_n \\ -\lambda & v_{n-1} - \lambda^2 \end{pmatrix}\begin{pmatrix} \varphi_n \\ -\varphi_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} v_n\varphi_n - \lambda v_n\varphi_{n-1} \\ -\lambda\varphi_n + (\lambda^2 - v_{n-1})\varphi_{n-1} \end{pmatrix} \end{aligned}$$

Hence we have the coupled equation

$$\begin{array}{l} \dot{\varphi}_n = v_n\varphi_n - \lambda v_n\varphi_{n-1} \\ \dot{\varphi}_{n-1} = \lambda\varphi_n - (\lambda^2 - v_{n-1})\varphi_{n-1} \end{array} \quad (7)$$

It follws from this coupled equation that we get the equality

$$v_n\varphi_n - \lambda v_n\varphi_{n-1} = \lambda\varphi_{n+1} - (\lambda^2 - v_n)\varphi_n. \quad (8)$$

Thus we have three term recurrences

$$\lambda\varphi_n = \varphi_{n+1} + v_n\varphi_{n-1}. \quad (9)$$

By (7) and (9),

$$\begin{aligned} \dot{\varphi}_n &= v_n\varphi_n - v_n\lambda\varphi_{n-1} \\ &= v_n\varphi_n - v_n(\varphi_n + v_{n-1}\varphi_{n-2}) \\ &= v_n\varphi_n - v_n\varphi_n - v_nv_{n-1}\varphi_{n-2} \\ &= -v_nv_{n-1}\varphi_{n-2} \quad (10) \end{aligned}$$

Thereby we obtain

$$\dot{\varphi}_n = -v_nv_{n-1}\varphi_{n-2} \quad (11)$$

Therefore we have the following result

**THEOREM** (**Vinet and Zhedanov (1998)**[8]) *If $\varphi_{-1} = 0$, $\varphi_0 = 1$ hold, then $\varphi_n$ is symmetric orthogonal polynomials.*

# 4 Negative Volterra equations and Cylinder functions

In this section we consider the relationship between special functions satisfying the identities

$$\lambda\psi_n(\lambda, t) = \psi_{n+1}(\lambda, t) + w_n(t)\psi_{n-1}(\lambda, t)$$

and negative Volterra equations. By using $\psi_n(\lambda, t)$ satisfying the identities, we construct Lax pair

$$\begin{cases} \Psi_{n+1} = U_n\Psi_n \\ \dot{\Psi}_n = V_n\Psi_n, \end{cases}$$

where we denote

$$\Psi_n = \begin{pmatrix} \psi_n(\lambda, t) \\ -\psi_{n-1}(\lambda, t) \end{pmatrix}$$

and denote

$$U_n = \begin{pmatrix} \lambda & w_n(t) \\ -1 & 0 \end{pmatrix}$$

$$V_n = \begin{pmatrix} \alpha_n^t(\lambda) & \beta_n^t(\lambda) \\ \gamma_n^t(\lambda) & \delta_n^t(\lambda) \end{pmatrix}$$

It follows form this pair that zero curvature representation

$$\frac{d}{dt}U_n = V_{n+1}U_n - U_nV_n$$

holds. From this representation we obtain two equations

$$\begin{cases} 0 = \lambda(\alpha_n^t(\lambda) - \alpha_{n+1}^t(\lambda)) - \gamma_{n+2}^t(\lambda)w_{n+1}(t) + w_n(t)\gamma_n^t(\lambda) \\ \dot{w}_n(t) = w_n(t)(\alpha_{n+1}^t(\lambda) - \alpha_{n-1}^t(\lambda) - (\gamma_n^t(\lambda) - \gamma_{n+1}^t(\lambda))\lambda). \end{cases} \quad (12)$$

If we let $\alpha_n^t(\lambda) = 0$, $\beta_n^t(\lambda) = -\dfrac{q_{n-1}(t)}{\lambda}$, $\gamma_n^t(\lambda) = \dfrac{1}{\lambda q_{n-1}(t)}$, $\delta_n^t(\lambda) = \dfrac{1}{q_{n-1}(t)}$, then we can get the equations

$$\begin{array}{l} \dot{w}_n = q_{n-1} - q_n \\ w_n = q_nq_{n-1}, \end{array} \quad (13)$$

where we denote as $q_n \equiv q_n(t)$ , $w_n = w_n(t)$ We call this equation "Negative Volterra equations". The reason we call "Negative " is that $\gamma_n$ depend on $\lambda^{-1}$. Now follows it from $\dot{\Psi}_n = V_n \Psi_n$ that we obtain

$$\begin{cases} \dot{\psi}_n = \frac{q_{n-1}}{\lambda}\psi_{n-1} \\ \dot{\psi}_{n-1} = -\frac{1}{\lambda q_{n-1}}\psi_n + \frac{1}{q_{n-1}}\psi_{n-1}. \end{cases} \quad (14)$$

**Remark.** $\{\psi_n\}$ is not symmetric orthogonal polynomials. We show this fact by proof of contradiction. Assume that $\{\psi_n\}$ is symmetric orthogonal polynomials. Then $\psi_{-1}(\lambda, t) = 0$ and $\psi_0(\lambda, t) = 1$ hold. Then we have $\dot{\psi}_1(\lambda, t) = \dfrac{q_0(t)}{\lambda}\psi_0(\lambda, t)$. Since $\psi_1(\lambda, t) = \lambda$ and $\lambda$ doesn't depend on $t$, we have $0 = \dfrac{q_0(t)}{\lambda} \cdot 1$. Namely $q_0(t) = 0$. Thus we obtain $w_1(t) = q_1(t)q_0(t) = q_1(t) \cdot 0 = 0$. This fact $w_1(t) = 0$ conflicts with **Favard's theorem**. Therefore it was shown that $\{\psi_n\}$ is not symmetric orthogonal polynomials.

Well we will consider what functions $\{\psi_n\}$ is. Since we can get $\psi_{n-1} = \dfrac{\lambda}{q_{n-1}}\dot{\psi}_n$ from (14), we obtain

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{\lambda}{q_{n-1}}\dot{\psi}_n\right) &= -\frac{1}{\lambda q_{n-1}}\psi_n + \frac{\lambda}{q_{n-1}^2}\dot{\psi}_n \\ \frac{-\lambda \dot{q}_{n-1}}{q_{n-1}^2}\dot{\psi}_n + \frac{\lambda}{q_{n-1}}\ddot{\psi}_n &= -\frac{1}{\lambda q_{n-1}}\psi_n + \frac{\lambda}{q_{n-1}^2}\dot{\psi}_n \\ -\frac{\dot{q}_{n-1}}{q_{n-1}}\dot{\psi}_n + \ddot{\psi}_n &= -\frac{1}{\lambda^2}\psi_n + \frac{1}{q_{n-1}}\dot{\psi}_n \end{aligned}$$

Therefore we have the ordinary differential equations

$$\ddot{\psi}_n - \frac{1 + \dot{q}_{n-1}}{q_{n-1}}\dot{\psi}_n + \frac{1}{\lambda^2}\psi_n = 0 \quad (15)$$

We let $q_n = \dfrac{t}{2n}$ which we can regard as the special solution for negative Volterra equations (13) and then we obtain ordinary diffrential equations

$$\ddot{\psi}_n - \frac{2n-1}{t}\dot{\psi}_n + \frac{1}{\lambda^2}\psi_n = 0, \quad (16)$$

Here we describe the solutions of this ordinary differential equations. At first we start from Bessel equations([2], [9]).

$$t^2\ddot{\mathscr{C}}_n(t) + t\dot{\mathscr{C}}_n(t) + (t^2 - n^2)\mathscr{C}_n(t) = 0$$

Here we put $\psi_n(\lambda, t) = t^{\beta n - \alpha}\mathscr{C}_n\left(\dfrac{t^\beta}{\lambda}\right)$. Then we obtain

$$\begin{aligned} \dot{\psi}_n &= (\beta n - \alpha)t^{\beta n - \alpha - 1}\mathscr{C}_n\left(\frac{t^\beta}{\lambda}\right) + t^{\beta n - \alpha}\dot{\mathscr{C}}_n\left(\frac{t^\beta}{\lambda}\right)\frac{\beta}{\lambda}t^{\beta - 1} \\ \dot{\psi}_n &= (\beta n - \alpha)t^{-1}\psi_n + \frac{\beta}{\lambda}t^{\beta(n+1) - \alpha - 1}\dot{\mathscr{C}}_n\left(\frac{t^\beta}{\lambda}\right) \end{aligned}$$

Thus we have

$$\dot{\mathscr{C}}_n\left(\frac{t^\beta}{\lambda}\right) = \frac{\lambda}{\beta}t^{-\beta(n+1)+\alpha+1}(\dot{\psi}_n - (\beta n - \alpha)t^{-1}\psi_n)$$

Moreover we differentiate this equality

$$\ddot{\mathscr{C}}_n\left(\frac{t^\beta}{\lambda}\right) = \left(\frac{\lambda}{\beta}\right)^2 t^{\alpha - 2\beta - \beta n + 1}\left\{t\ddot{\psi}_n + (1 + 2\alpha - 2\beta n - \beta)\dot{\psi}_n + (\alpha - \beta n)(\alpha - \beta n - \beta)t^{-1}\psi_n\right\}$$

On the eqaution

$$t^2\ddot{\mathscr{C}}_n(t) + t\dot{\mathscr{C}}_n(t) + (t^2 - n^2)\mathscr{C}_n(t) = 0$$

we substitute $\dfrac{t^\beta}{\lambda}$ for $t$, then we obtain

$$\frac{t^{2\beta}}{\lambda^2}\ddot{\mathscr{C}}_n\left(\frac{t^\beta}{\lambda}\right) + \frac{t^\beta}{\lambda}\dot{\mathscr{C}}_n\left(\frac{t^\beta}{\lambda}\right) + \left(\frac{t^{2\beta}}{\lambda^2} - n^2\right)\mathscr{C}_n\left(\frac{t^\beta}{\lambda}\right) = 0$$

Substituting $\psi_n$ for $\mathscr{C}_n$, we obtain

$$\ddot{\psi}_n - \frac{2(\beta n - \alpha) - 1}{t}\dot{\psi}_n + \left(\frac{1}{\lambda^2}\beta^2 t^{2\beta} + \alpha(\alpha - 2\beta n)\right) t^{-2}\psi_n = 0$$

We let $\alpha = 0$ and $\beta = 1$, then we get the equations

$$\ddot{\psi}_n - \frac{2n-1}{t}\dot{\psi}_n + \frac{1}{\lambda^2}\psi_n = 0$$

**Recurrence formulae for cylinder functions**

Following Watson[9], we have two recurrences for *cylinder functions* $\mathscr{C}$

$$\begin{aligned} \mathscr{C}_{n-1}(t) + \mathscr{C}_{n+1}(t) &= \frac{2n}{t}\mathscr{C}_n(t) \\ \mathscr{C}_{n-1}(t) - \mathscr{C}_{n+1}(t) &= 2\dot{\mathscr{C}}_n(t) \end{aligned}$$

At first we consider the recurrence $\mathscr{C}_{n-1}(t) + \mathscr{C}_{n+1}(t) = \frac{2n}{t}\mathscr{C}_n(t)$. We substitute $\dfrac{t}{\lambda}$ for $t$, then we can obtain

$$\mathscr{C}_{n-1}\left(\frac{t}{\lambda}\right) + \mathscr{C}_{n+1}\left(\frac{t}{\lambda}\right) = \frac{2n}{t}\lambda\mathscr{C}_n\left(\frac{t}{\lambda}\right).$$

Multiplying both sides by $t^{n+1}$,

$$t^{n+1}\mathscr{C}_{n-1}\left(\frac{t}{\lambda}\right) + t^{n+1}\mathscr{C}_{n+1}\left(\frac{t}{\lambda}\right) = t^n 2n\lambda\mathscr{C}_n\left(\frac{t}{\lambda}\right).$$

hold. Here we put $\psi_n^t(\lambda) \equiv t^n\mathscr{C}_n\left(\dfrac{t}{\lambda}\right)$, we can get the recurrence

$$t^2\psi_{n-1}^t(\lambda) + \psi_{n+1}^t(\lambda) = 2n\lambda\psi_n^t(\lambda). \tag{17}$$

Moreover put $\psi_n^t(\lambda) \equiv (2n-2)(2n-4)(2n-6)\cdots 4\cdot 2\mathscr{P}_n^t(\lambda)$, we have the identities

$$t^2\left(\prod_{j=1}^{n-2} 2j\right)\mathscr{P}_{n-1}^t(\lambda) + \left(\prod_{j=1}^{n} 2j\right)\mathscr{P}_{n+1}^t(\lambda) = 2n\lambda\left(\prod_{j=1}^{n-1} 2j\right)\mathscr{P}_n^t(\lambda)$$

and

$$t^2\mathscr{P}_{n-1}^t(\lambda) + 2n(2n-2)\mathscr{P}_{n+1}^t(\lambda) = 2n(2n-2)\lambda\mathscr{P}_n^t(\lambda)$$

hold. Thus we can get

$$\lambda\mathscr{P}_n^t(\lambda) = \mathscr{P}_{n+1}^t(\lambda) + \frac{t^2}{2n(2n-2)}\mathscr{P}_{n-1}^t(\lambda),$$

excluding $n = 0$ and $n = 1$. Next we deform $\mathscr{C}_{n-1}(t) - \mathscr{C}_{n+1}(t) = 2\dot{\mathscr{C}}_n(t)$. We substitute $\dfrac{\lambda}{t^n}\dot{\psi}_n^t(\lambda) - \dfrac{\lambda n}{t^{n+1}}\psi_n^t(\lambda)$ for $\dot{\mathscr{C}}_n\left(\dfrac{t}{\lambda}\right)$ on this recurrence, we have

$$2\left(\frac{\lambda}{t^n}\dot{\psi}_n^t(\lambda) - \frac{\lambda n}{t^{n+1}}\psi_n^t(\lambda)\right) = \mathscr{C}_{n-1}\left(\frac{t}{\lambda}\right) - \mathscr{C}_{n+1}\left(\frac{t}{\lambda}\right)$$

Multiplying bothside by $t^n$,

$$2\left(\lambda\dot{\psi}_n^t(\lambda) - \frac{\lambda n}{t}\psi_n^t(\lambda)\right) = t\cdot t^{n-1}\mathscr{C}_{n-1}\left(\frac{t}{\lambda}\right) - t^{-1}t^{n+1}\mathscr{C}_{n+1}\left(\frac{t}{\lambda}\right)$$

$$2\lambda\dot{\psi}^t(\lambda) - \frac{2\lambda n}{t}\psi_n^t(\lambda) = t\psi_n^t(\lambda) - t^{-1}\psi_{n+1}^t(\lambda) \tag{18}$$

It follows from (17) and (18) that we have

$$\begin{aligned} 2\lambda\dot{\psi}_n^t(\lambda) - \frac{t^2\psi_{n-1}^t(\lambda) + \psi^t(\lambda)}{t} &= t\psi_{n-1}^t(\lambda) - t^{-1}\psi_{n+1}^t(\lambda) \\ 2\lambda\dot{\psi}_n^t(\lambda) - t\psi_{n-1}^t(\lambda) - t^{-1}\psi_{n+1}^t(\lambda) &= t\psi_{n-1}^t(\lambda) - t^{-1}\psi_{n+1}^t(\lambda) \\ 2\lambda\dot{\psi}_n^t(\lambda) &= 2t\psi_n^t(\lambda) \\ \dot{\psi}_n^t(\lambda) &= \frac{t}{\lambda}\psi_{n-1}^t(\lambda) \end{aligned}$$

Multiplying both side by $\frac{1}{2^{n-1}(n-1)!}$,

$$\frac{\dot{\psi}^t_n(\lambda)}{2^{n-1}(n-1)!} = \frac{t}{\lambda}\frac{\psi^t_{n-1}(\lambda)}{2^{n-1}(n-1)!}$$

Therefore we obtain

$$\dot{\mathscr{P}}^t_n(\lambda) = \frac{t}{2(n-1)\lambda}\mathscr{P}^t_{n-1}(\lambda)$$

Moreover it follows from the equations

$$\ddot{\psi}_n + \frac{2n-1}{t}\dot{\psi}_n + \frac{1}{\lambda^2}\psi_n = 0$$

that we have

$$\frac{\partial^2}{\partial t^2}\left(\frac{\psi_n}{2^{n-1}(n-1)!}\right) - \frac{2n-1}{t}\frac{\partial}{\partial t}\left(\frac{\psi_n}{2^{n-1}(n-1)!}\right) + \frac{1}{\lambda^2}\left(\frac{\psi_n}{2^{n-1}(n-1)!}\right) = 0.$$

Therefore we can obtain

$$\frac{\partial^2}{\partial t^2}\mathscr{P}^t_n(\lambda) - \frac{2n-1}{t}\frac{\partial}{\partial t}\mathscr{P}^t_n(\lambda) + \frac{1}{\lambda^2}\mathscr{P}^t_n(\lambda) = 0.$$

Hence we have the following result

**PROPOSITION** *If we have two recurrences and one second order differential equations*

$$\begin{aligned}\lambda\mathscr{P}^t_n(\lambda) &= \mathscr{P}^t_{n+1}(\lambda) + \frac{t^2}{2n(2n-2)}\mathscr{P}^t_{n-1}(\lambda)\\ \dot{\mathscr{P}}^t_n(\lambda) &= \frac{t}{2(n-1)\lambda}\mathscr{P}^t_{n-1}(\lambda)\end{aligned}$$

$$\frac{\partial^2}{\partial t^2}\mathscr{P}^t_n(\lambda) - \frac{2n-1}{t}\frac{\partial}{\partial t}\mathscr{P}^t_n(\lambda) + \frac{1}{\lambda^2}\mathscr{P}^t_n(\lambda) = 0,$$

*excluding* $n = 0$ *and* $n = 1$, *then we obtain the cylinder functions*

$$\mathscr{P}^t_n(\lambda) = \frac{t^n}{2^{n-1}(n-1)!}\mathscr{C}_n\left(\frac{t}{\lambda}\right).$$

# 5 Conclusion

In this paper we demonstrated special functions related to Lotka-Volterra equations and negative Volterra equations introduced from zero curvature representations. We showed that there is the relationships between Lotka-Volterra equations and symmetric orthogonal polynomials, and the relationships between negative Volterra equations with special solutions and cylinder functions. The relationships between negative Volterra equations and cylinder functions give us establish a new property of Bessel functions.

On future issues, our view is demonstarating a relationships between *discrete* negative Volterra equations

$$u^{(m+1)}_n - u^{(m)}_n = \delta\left(q^{(m)}_{n-1} - q^{(m+1)}_n\right) \qquad u^{(m)}_n = q^{(m)}_n q^{(m)}_{n-1}$$

introduced by using Hirota's direct method and special functions, where we denote $n$, $m$ as a integer number. It is well-known that we can obtain the discrete Lotka- Volterra equations

$$D^{(m+1)}_n\left(\lambda^{(m+1)} - D^{(m+1)}_{n-1}\right) = D^{(m)}_n\left(\lambda^{(m)} - D^{(m)}_{n+1}\right)$$

from the spectral transfomation and three term recurrences for symmetric orthogonal polynomials[6]. But we do not know special functions connected to discrete negative Volterra equations. Thus we will discover the relationships between discrete negative Volterra equations and special functions.

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