

A Modified Fixed Point Method for the Perona Malik Equation

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Abstract: In this work we present a new method to solve the Perona Malik equation for the image denoising. The method is based on a modified fixed point algorithm which is fast and stable. We discretize the equation using a finite volume method by integrating the equation using a fuzzy measure on the control volume. To make our algorithm move faster in time, we have used an optimized domain decomposition which generalize the wave relaxation method. Several test of noised images illustrate this approach and show the efficiency of the proposed new method.

Key words: Perona Malik equation, Fixed point method, Fuzzy measure, Choquet integral.

1 Introduction

Perona and Malik [2] invented in 1990 for image filtering, the model:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(g(\|\nabla u\|)\nabla u) = 0 & \text{on } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times [0, T] \\ u = h_0 & \text{on } t = 0 \end{cases} \quad (1)$$

Where Ω is an image domain in \mathbb{R}^2 and g (called edge stopping function) is a positive function chosen to satisfy $g(0) = 1$ and $\lim_{x \rightarrow +\infty} g(x) = 0$. h_0 is the grey level distribution of a given distorted image. Typical example for an edge stopping function g which have been used in fact by Perona and Malik, are:

$$g(s) = \frac{1}{1 + s^2}, \quad g(s) = e^{-s} \quad s \geq 0$$

The experimental results obtained by Malik and Perona are perceptually impressive and show that an edge detector based on this theory gives edges which remain much more stable across the scales. However, this model had serious practical and difficulties which have been sought by many researchers in the last decades [1],.. [7]. The first difficulty was that for large white noise, the smoothing doesn't give good result because the gradient of the image will be unbounded. The second difficulty arose from the equation itself; for positive function g , there is no correct theory of equation (1). In [1] we prove that the resolution of the equation is instable and the resultant edge is inaccurate. One solution to this problem is to make some modification to the equation. Since the Perona and Malik, a considerable research was devoted to the understanding and the stability and the theory of this equation. The pioneer solutions to this problems we can found it in [1], [5], [6]. Noting that all the solutions proposed to the Perona and Malik issue was about modifying the equation to have stability of the edge and the solution. another solution

was using variational studies to enhance the equation . One problem remain from this method: Because of smoothing process, Nearest grey levels tend to the same grey level, so the picture lose information. Also the image become blurred. Here in this paper, we study generally the existence of a solution to the problem and new robust, accurate, fast numerical methods to solve this equation.

We first describe the fixed point to linearize the implicit scheme of Equation 1, We prove taking account some condition, the existence of a solution of the variational problem resulting from it, then we propose a new modified fixed point to converge faster to the solution.

Secondly,the light is a non additive variable, the sum of too light is not eventually the additive sum of the the two light functions: a high grey level added to a high grey level is not necessary a black level. You can see the reference [8,12 an 13] to know that the light is not additive. For this reason using a classical finite volume or finite difference is not consistent. Thus we proposed to use a fuzzy measure instead of Lebesgue measure to take into account the non additivity of the light. We use a Choquet integral to approach the equation in control volumes.

We use and generalize an optimized domain decomposition to reduce the cost of time of our algorithm. The domain decomposition method (OO2) is a tool we use for large domain sizes and thin meshes, it consists of solving in parallel our equations on sub domains with new interface condition in order to reduce the size and the complexity of the problem, the parallelism in new devices of computer science and architecture make this method easy to handle. There is so many kind of methods in the domain decomposition method (see [27] and [28] for some classical ones), but in this paper we consider an optimized domain decomposition method of two order (OO2). Severally used this last decades ([20],...[24]), this method is powerful and optimized. It's optimized because ([20],[21]) we can compute explicitly the rate of convergence between two iteration using the Fourier transform and make this rate optimal so the method will converge quickly. But, the rate of convergence of this method can be computed only for linear partial differential equation, and it's difficult to have and explicit rate for nonlinear partial differential equation, Thus come the idea of the modified fixed method combined to this optimized fixed point to solve this problem of nonlinearity.

At the end of this work you can find some numerical experiment that we can compare to previous method and that effectively illustrate the efficiency of our proposed methods.

2 Method of Modified Fixed point

In this section we present the Fixed point algorithm and the modified one to solve the stationary problem 1

By an implicit Scheme of discretization, and some changes, the equation 1 could be made as:

$$\begin{cases} cu - \operatorname{div} g(u)\nabla u &= f(x, y) \text{ on } \Omega \\ u &= h \text{ on } \partial\Omega \end{cases} \quad (2)$$

We could also consider $x \geq 1$ and $y \geq 1 \forall (x, y) \in \Omega$

The Fixed point method involves given one initial function u_0 , we construct iteratively a function sequence u_n as follows:

$$\begin{cases} cu_{n+1} - \operatorname{div} g(u_n)\nabla u_{n+1} &= f(x, y) \text{ on } \Omega \\ u_{n+1} &= h \text{ on } \partial\Omega \end{cases} \quad (3)$$

So we give below a convergence result of this algorithm:

Theorem 2.1 *Suppose that:*

- $\exists \nu > 0$ and $\mu; c \geq 0$ and $g(x) > \nu$
- g is K -lipshitzian.
- $\frac{K}{\nu} < 1$

Then the sequence $u_{n+1} = \phi(u_n)$ converge to the unique solution of the nonlinear problem (3)

Proof:

By an adequate change, it is sufficient to consider a proof with the Dirichlet condition $u = 0$ (The same thing could be done with a Neumann condition, considering specific spaces and inequalities). Let $V = \{u \in H_0^1(\Omega) / g(u) \in L^1(\Omega) \text{ and } \|\nabla u\| < M\}$. Consider the following application:

$$\begin{aligned} \phi : V &\mapsto V \\ v &\mapsto u \end{aligned}$$

such that u is the unique solution of the variational formulation:

$$\int_{\Omega} cu w + \int_{\Omega} g(v) \nabla u \nabla w = \int_{\Omega} f(x, y) w \quad \forall w \in H_0^1(\Omega) \quad (4)$$

♣ ϕ bellow is well defined, Indeed: Consider the bilinear form :

$$a(u, w) = \int_{\Omega} cu w + \int_{\Omega} g(u) \nabla u \nabla w$$

by the Holder and the Poincarré Inequality a is continuous:

$$\begin{aligned} a(u, w) &\leq \sup_{\Omega} |c| \|u\| \|w\| + M \|\nabla u\| \|\nabla w\| \\ &\leq (\sup_{\Omega} |c| C_{Poincarre}^2) + M \|\nabla u\| \|\nabla w\| \end{aligned}$$

a is coercive:

$$\begin{aligned} a(u, u) &= \int_{\Omega} cu u + \int_{\Omega} g(u) \nabla u \nabla u \\ &\geq \nu \|u\|^2 \end{aligned}$$

The Lax Milgram theorem involves that there exist one solution of (3) named $\phi(v)$

♣ we have by substraction:

$$a(\phi(u) - \phi(v), w) = 0 \quad \forall w \in H_0^1(\Omega)$$

in another way

$$\int_{\Omega} c(\phi(u) - \phi(v)) w + \int_{\Omega} g(u) \nabla(\phi(u) - \phi(v)) \nabla w = \int_{\Omega} \nabla \phi(v) (g(v) - g(u))$$

We take $w = \phi(u) - \phi(v)$, so (Thinks to holder and Poincarre Inequalities)

$$\nu \|\phi(u) - \phi(v)\|^2 \leq K \times M \times C_{Poincarre} \|\phi(u) - \phi(v)\| \|u - v\|$$

M could be chosen as $M \times C_{Poincarre} < 1$, Thus

$$\|\phi(u) - \phi(v)\| \leq \frac{K}{\nu} \|u - v\|$$

The theorem of fixed point applied to the application ϕ show that the equation $\phi(u) = u$ have one solution and the suite $u_{n+1} = \phi(u_n)$ converge to this solution which is the solution of problem (4)

Remark 1 In the proof of the theorem, it was obliged to take M small, but in general M must be greater than $\max_{\Omega} \nabla h_0$, but this condition is not true in general. so the next theorem try to give a solution to this issue.

Remark 2 In the proof of the theorem we have:

$$\nu \|\phi(u) - \phi(v)\| \leq K \|u - v\|$$

If K is smaller, the Fixed point suite converge quickly. So we introduce a new modified fixed point by adding a new function r such that:

$$\begin{cases} cu_{n+1} - \operatorname{div}((g(u_n) + \mathbf{r}(\mathbf{u}_n, \nabla \mathbf{u}_n)) \nabla u_{n+1}) &= f(x, y) - \operatorname{div}(\mathbf{r}(\mathbf{u}_n, \nabla \mathbf{u}_n) \nabla u_n) \text{ on } \Omega \\ u_{n+1} &= o \text{ on } \partial\Omega \end{cases} \quad (5)$$

r is a function we choose to have the Coefficient K_r small. It's necessary to have $r : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ that assume

$$|g'(x) + r'(x, \vec{y})| < \varepsilon_1 \ll 1 \text{ and } \|\nabla_y(r(x, \vec{y}))\| < \varepsilon_2 \ll 1$$

Using r , the condition $g(x) > \nu$ also we don't have to look for a solution with small gradient shown in theorem 1.1 proof.

Theorem 2.2 Suppose that g verify: $i - 0 \leq g(x) \leq 1$ and $ii - 0 \leq g'(x) \leq 1$

Then, there exist a function r such that:

$$a) \quad \nu < g(x) + r(x, \vec{y})$$

$$b) \quad |g'(x) + r'(x, \vec{y})| < \varepsilon_1 \ll \frac{1}{2}$$

$$c) \quad \|\nabla_y(r(x, \vec{y}))\| < \varepsilon_2 \ll \frac{1}{2}$$

and the sequence $u_{n+1} = \phi(u_n)$ of the modified fixed point converge to a solution of the nonlinear problem (3)

Proof: We take r such that:

$$r(x, \vec{y}) = -g(x) \times \exp(-\|\vec{y}\| \times \varepsilon_1^2) + \frac{1}{\varepsilon_2} \exp(-(1 - \varepsilon_2) \ln(\|\vec{y}\|))$$

that function verify of course, proprieties a), b) and c).

Let $V = \{u \in H_0^1(\Omega) / g(u) \text{ and } r(u) \in L^1(\Omega) \text{ and } \|\nabla u\| < M\}$.

Consider the following application:

$$\begin{aligned} \phi : V &\mapsto V \\ v &\mapsto u \end{aligned}$$

such that u is the unique solution of the variational formulation:

$$\int_{\Omega} cuw + \int_{\Omega} (g(v) + r(v)) \nabla u \nabla w = \int_{\Omega} fw + \int_{\Omega} r(v) \nabla v \nabla w \quad \forall w \in H_0^1(\Omega) \quad (6)$$

♣ ϕ is well defined, Indeed: Consider the bilinear form :

$$a(u, w) = \int_{\Omega} cuw + \int_{\Omega} (g(v) + r(v)) \nabla u \nabla w$$

by the Holder and the Poincaré Inequality a is continuous:

$$\begin{aligned} a(u, w) &\leq \sup_{\Omega} |c| \|u\| \|w\| + \mu \|\nabla u\| \|\nabla w\| \\ &\leq (\sup_{\Omega} |c| C_{Poincarre}^2 + \mu) \|\nabla u\| \|\nabla w\| \end{aligned}$$

a is coercive:

$$a(u, u) = \int_{\Omega} cuu + \int_{\Omega} g(u)\nabla u \nabla u \geq \nu \|u\|^2$$

The Lax Milgram theorem involves that there exist one solution of (6) named $\phi(v)$

♣ It's easy to make:

$$\begin{aligned} & \int_{\Omega} c|\phi(u) - \phi(v)|^2 + \int_{\Omega} (g(u) + r(u)) |\nabla(\phi(u) - \phi(v))|^2 = \\ & = \int_{\Omega} (g(v) + r(v) - g(u) - r(u))\nabla\phi(v)\nabla(\phi(u) - \phi(v)) + \\ & \quad + \int_{\Omega} (r(u)\nabla u - r(v)\nabla v)\nabla(\phi(u) - \phi(v)) \end{aligned}$$

so

$$\begin{aligned} \nu \int_{\Omega} \|\nabla(\phi(u) - \phi(v))\|^2 & \leq \varepsilon_1 M \|u - v\| \|\nabla(\phi(u) - \phi(v))\| + \varepsilon_2 M \|u - v\| \|\nabla(\phi(u) - \phi(v))\| \\ & \leq C_{Poincarre}(M + 1)(\varepsilon_1 + \varepsilon_2) \|u - v\| \|\nabla(\phi(u) - \phi(v))\| \end{aligned}$$

We choose M such that $\frac{C_{Poincarre}(M+1)}{\nu} < 1$ The theorem of fixed point applied to the application ϕ show that the equation $\phi(u) = u$ have one solution and the suite $u_{n+1} = \phi(u_n)$ converge to this solution which is the solution of the variational problem of (2)

Notice that the function g in the Perona Malik model verify properties i and ii.

3 The finite volume method

Notice that when we use a finite difference or a finite volume method to approach equation 2, we can see that we process consecutively by computing a weighted average of the pixels around a single pixel. As we told in the introduction, the light is not additive so this average is not accurate. So it comes the idea to use a fuzzy measure and integrals to approach this average giving an importance to the element surrounding a pixel regarding its role in the control volume containing the pixels. We define the fuzzy measure as follows:

Definition 3.1 a Fuzzy measure μ is an application such that:

$$A \subset B \Rightarrow \mu(A) \leq \mu(B)$$

$$\mu(\emptyset) = 0$$

The Choquet integral on the control volume of a function f is:

$$OF(f)(i) = \sum_{x_i \in C} (f(x_i) - f(x_{i-1}))\mu(A_i)$$

where

$$A_i = \{x_i, x_{i+1}, \dots, x_n\}$$

The Choquet integral is an integral that generalize the Lebesgue integral and have many application in signal and image analysis. For more information about Choquet integrals I cite [15]. To realize the image filtering we take:

$$\mu_i = \min(1, (1 - \gamma_i)\sigma)$$

$$, \sigma > 0 \quad \gamma_i = \frac{\sum_{I_j \in C} |I_i - I_j| \alpha_{i,j}^\beta}{\sum_{j=1}^N \alpha_{i,j}^\beta}$$

$$\text{and } \alpha_{i,j}^\beta = \frac{1}{1+d(i,j)}$$

I_i is the image grey level on the pixel i . and $d(i, j)$ is the distance between i and j . (C the control volume) The we apply this integral to approach each term of equation 2.

Notice that there is other ways to choose the withes μ_i for filtering. One who is interested in this choice can see [16]

There are other choices of the g_i ([9,10,11]) which gives good results.

4 Domain decomposition with optimized interface of second order(DDM OO2)

The use of finite volumes , finite differences or finite elements solvers on high order meshes requires a high cost of computation.

Domain decomposition methods can reduce this cost by splitting initial problem into two or more sub-problems with smaller dimensions. Many authors have studied domain decomposition methods these last decades [17, 18]. Among these methods we consider in this work the method called second order optimized method OO2. This method was developed by different authors [20,21]. Our stationary equation (2) could be treated in what follow as a reaction advection diffusion equation (the same equation indeed). The main idea of the OO2 technique is described briefly as follows:

We split the domain Ω for example in two sub-domains Ω_1 and Ω_2 with an interface Γ (see figure 1) then we built two sequences $u_{1,p}^n$ and $u_{2,p}^n$ respectively solutions of two sub-problems as described

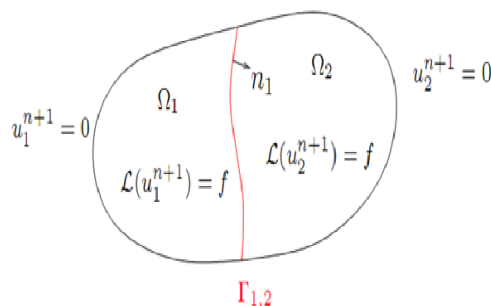


Figure 1: splitting of the domain in two sub-domain

bellow:

we choose two initials functions u_1^0 defined on Ω_1 and u_2^0 defined on Ω_2 then we consider the two problems:

$$\begin{cases} L(u_1^{p+1}) = f(x, y) \text{ on } \Omega_1 \\ u_1^{p+1} = 0 \text{ on } \partial\Omega \\ B_1(u_1^{p+1}) = B_1(u_2^p) \text{ on } \Gamma \end{cases} \quad (7)$$

and

$$\begin{cases} L(u_2^{p+1}) = f(x, y) \text{ on } \Omega_2 \\ u_2^{p+1} = 0 \text{ on } \partial\Omega \\ B_2(u_2^{p+1}) = B_2(u_1^p) \text{ on } \Gamma \end{cases} \quad (8)$$

Where

$$L(u) = cu + a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} - \mu \Delta u \quad (9)$$

$$\begin{aligned} B_1(u) &= \frac{\partial u}{\partial n} - C_1 u + C_2 \frac{\partial u}{\partial \tau} - C_3 \frac{\partial^2 u}{\partial \tau^2} \\ B_2(u) &= -\frac{\partial u}{\partial n} - (C_1 - \frac{a}{\mu})u + C_2 \frac{\partial u}{\partial \tau} - C_3 \frac{\partial^2 u}{\partial \tau^2} \end{aligned} \quad (10)$$

n and τ are the normal and the tangent on Ω_1

Because of the Fourier analysis we show that the rate of convergence in the fourier way is (see [6] for proof)

$$\rho(C_1, C_2, C_3, k) = \left(\frac{\lambda^-(k) - C_1 + ikC_2 + C_3k^2}{\lambda^+(k) - C_1 + ikC_2 + C_3k^2} \right)^2 \quad (11)$$

where

$$\lambda^\mp(k) = \frac{a^\mp \sqrt{a^2 + 4c\mu - 4i\mu bk + 4k^2\mu^2}}{2\mu}$$

The next theorem gives a condition of convergence of the OO2 method

Theorem 4.1 *Suppose that $c > 0$ and $\text{sign}(b) = \text{sign}(C_2)$ and $C_3 \geq 0$ then,*

$$\max_{|k| < \frac{\pi}{h}} \|\rho(C_1, C_2, C_3, k)\| < 1$$

Proof. see [21]

Remark 3 *In order to optimize the method we need to optimize the rate of convergence so we look for:*

$$\min_{C_1, C_2, C_3} \max_{|k| < \frac{\pi}{h}} \|\rho(C_1, C_2, C_3, k)\|$$

For optimizing this rate we have implemented the global optimization method [25]. Notice that this last problem have at least two optimums.

In our work, the viscosity is high and for the OO2 method proposed the convergence take more time because the rate of convergence is not small enough and that can be seen experimentally on so much example of high viscosity, also we can prove that theoretically. Thus, to have a convergence near to two iterations, we take the generalized artificial coefficients:

$$\begin{aligned} B_1(u) &= \frac{\partial u}{\partial n} - C_1 u + C'_1 u(0, y - a) + C_2 \frac{\partial u}{\partial \tau} + C'_2 \frac{\partial u(0, y - a)}{\partial \tau} \\ &\quad - C_3 \frac{\partial^2 u}{\partial \tau^2} + C'_3 \frac{\partial^2 u(0, y - a)}{\partial \tau^2} + C_4 \frac{\partial^2 u}{\partial n \partial \tau} \\ B_2(u) &= -\frac{\partial u}{\partial n} - C_1 u + C'_1 u(0, y - a) + C_2 \frac{\partial u}{\partial \tau} + C'_2 \frac{\partial u(0, y - a)}{\partial \tau} \\ &\quad - C_3 \frac{\partial^2 u}{\partial \tau^2} + C'_3 \frac{\partial^2 u(0, y - a)}{\partial \tau^2} + C_4 \frac{\partial^2 u}{\partial n \partial \tau} \end{aligned}$$

The fourier transform of the the rate of convergence is calculated using:

$$\begin{aligned} \mathfrak{F}(f(x - a)) &= e^{-ika} \mathfrak{F}(f(x)) \\ \mathfrak{F}\left(\frac{\partial^2 f}{\partial x \partial y}\right) &= ik \mathfrak{F}\left(\frac{\partial f}{\partial x}\right) \end{aligned}$$

We also optimize the step of our algorithm by Coupling OO2 and Fixed point next subproblems:

$$\begin{cases} L(u_1^{p+1}) = f(x, y) + \text{div}(r(u_1^p)\nabla u_1^p) & \text{on } \Omega_1 \\ u_1^{p+1} = 0 & \text{on } \partial\Omega \\ B_1(u_1^{p+1}) = B_1(u_2^p) & \text{on } \Gamma \end{cases} \quad (12)$$

and

$$\begin{cases} L(u_2^{p+1}) = f(x, y) + \text{div}(r(u_1^p)\nabla u_2^p) & \text{on } \Omega_2 \\ u_2^{p+1} = 0 & \text{on } \partial\Omega \\ B_2(u_2^{p+1}) = B_2(u_1^p) & \text{on } \Gamma \end{cases} \quad (13)$$

where:

$$L(u) = cu + \text{div}((r(u, \|\nabla u\|) + g(u))\nabla u) \quad (14)$$

5 Numerical simulation

In what follow we give some image filtered by the proposed method. We implemented the fixed points and finite volume in the Matlab software. The obtained images are all closes to the original ones and have a significant entropy and contrast.

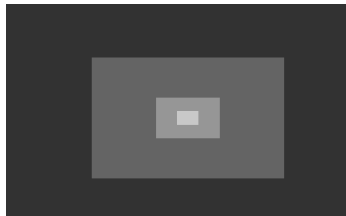


Figure 2: Original image

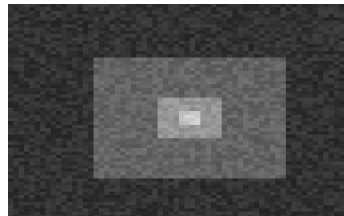


Figure 3: noised image

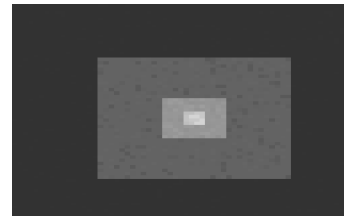


Figure 4: Filtered image

Here the image is noised with a 20 % gaussian

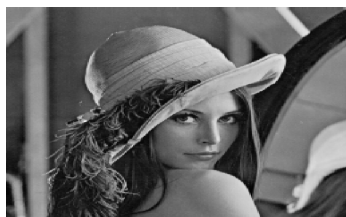


Figure 5: Original image



Figure 6: noised image



Figure 7: Filtered image

Here the image is noised with 10 % gaussian noise

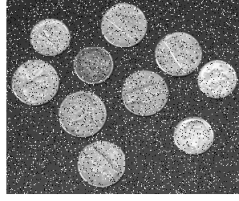


Figure 8: Original image

Figure 9: noised image

Figure 10: Filtered image

This image is noised with 30 % salt-pepper noise. We can see that in the denoised image that the shape in the coins is conserved contrary to the image obtained by the classical Perona Malik methods.

By our method, the RMS at t=40 iterations image was $6.7 * 10^{-4}$ for our method for a 10% noisy image (eventually the lena image in figure 4). The Entropy was about 1.01

The RMS at t=60 iterations r the was 0.03 for our method for a 30% noisy lena image. The Entropy was about 3.47.

The RMS between two images A and B is:

$$RMS = \frac{1}{MN} \sum_{i=1}^N \sum_{j=1}^M (A(i, j) - B(i, j))^2$$

The necessary time for the successive computation of the algorithm and using the optimized decomposition method is about 18s while a direct computation take a lot of time.

6 Conclusion

In this work we have developed an new fast and efficient algorithm for image filtering based on the Perona Malik equation. We firstly have given a proof of the convergence of the modified fixed point technique applied to the non linear equation that we proposed. We have proposed a new robust finite volume based on the Choquet integral to solve the Perona Malik equation. Then we have used a generalized the OO2 domain decomposition method in order to make our algorithm faster. Secondly we have presented several test-cases to show the efficiency of this approach. The fundamental result is that we obtained a fast and efficient filter for noised images. As perspective of the present work, we can study the following ideas:

- to prove the convergence of the proposing finite volume and using it for other physical area. -
- To use an adaptive and specific control volume to solve the equation .
- Apply the method to real problems in fluid dynamics and environmental science.

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