Relativistic Treatment of Massive Klein-Gordon Particle by Modified Generalized Hulthen Potential

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Abstract: Approximate bound state solutions of spinless particles with a special case of equal scalar and vector modified generalized Hulthen potential has been obtained under the massive Klein-Gordon equation. The energy eigenvalues and the corresponding wave functions expressed in terms of a Jacobi polynomial are also obtained using the parametric generalization of the Nikiforov-Uvarov (NU) method. Under limiting cases our result are in agreement with the existing literature. Our results could be used to study the interactions and binding energies of the central potential for diatomic molecules in the relativistic framework which have many applications in physics and some others related disciplines.

Keywords: Massive Klein-Gordon equation, modified generalized Hulthen potential, eigenvalues, wave function, Nikiforov-Uvarov method and potential barrier.

1. Introduction

The studies of exact solutions of the quantum mechanical wave equations such as Schrödinger, Klein-Gordon, Dirac, Salpeter etc. have great importance in atomic and molecular physics and have attracted much attention and interest since the early development of quantum mechanics till today [1-13]. In nuclear and high energy physics, one of the interesting problems is to obtain exact solution of the Klein-Gordon and Dirac equations. When a particle is in a strong potential field, the relativistic effect must be considered, which gives the correction for non relativistic quantum mechanics [14]. In solving non relativistic or relativistic wave equation whether for central or non central potential, various methods are used such as Asymptotic iteration method (AIM) [15], Super symmetric quantum mechanics (SUSYQM) [16] shifted $\frac{1}{N}$ expansion [17], factorization [18], Nikiforov-Uvarov (NU)

method [19] etc.

In the relativistic quantum mechanics, one can apply the Klein-Gordon equation to the treatment of a zero-spin particle. Recently, many studies have been carried out to explore the relativistic energy eigenvalues and corresponding wave functions of the Klein-Gordon and Dirac equations [14, 20, 21]. The aim of this paper is to obtain the energy eigenvlaues and the corresponding eigen functions for the massive Klein-Gordon particle under modified generalized Hulthen potential in a case of equal scalar and vector using the parametric generalization of the Nikiforov-Uvarov (NU) method.

2. Brief Review of Nikiforov-Uvarov (NU) Method

The conventional NU method was presented by Nikiforov and Uvarov [19] and has been employed to solve second order differential equations such as the Schrödinger, Klein-Gordon and Dirac equations etc. The parametric generalization of the NU method is given by the generalized hyper-geometric type equation as [27]

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$$\Psi''(s) + \frac{(c_1 - c_2 s)}{s(1 - c_3 s)} \Psi'(s) +$$

$$\frac{1}{s^2 (1 - c_3 s)^2} \left[-\xi_1 s^2 + \xi_2 s - \xi_3 \right] \Psi(s) = 0 \tag{1}$$

According to the NU method, the energy eigenvalues equation and eigen functions, respectively, satisfy the following sets of equations

$$c_2 n - (2n+1)c_5 + (2n+1)\left(\sqrt{c_9} + c_3\sqrt{c_8}\right) +$$

$$n(n-1)c_3 + c_7 + 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0, \tag{2}$$

$$\psi(s) =$$

$$N_{nl} s^{c_{12}} (1 - c_3 s)^{-c_{12} - (c_{13}/c_3)} P_n^{\left(c_{10} - 1, \frac{c_{11}}{c_3} - c_{10} - 1\right)} (1 - 2c_3 s), \tag{3}$$

where

$$c_4 = \frac{1}{2}(1 - c_1), \, c_5 = \frac{1}{2}(c_2 - 2c_3),$$

$$c_6 = c_5^2 + \xi_1, \, c_7 = 2c_4 c_5 - \xi_2, \, c_8 = c_4^2 + \xi_3,$$

$$c_9 = c_3 c_7 + c_3^2 c_8 + c_6, \, c_{10} = c_1 + 2c_4 + 2\sqrt{c_8},$$

$$c_{11} = c_2 - 2c_5 + 2\left(\sqrt{c_9} + c_3\sqrt{c_8}\right), \, c_{12} = c_4 + \sqrt{c_8},$$

$$c_{13} = c_5 - \left(\sqrt{c_9} + c_3\sqrt{c_8}\right) \tag{4}$$

and P_n is the orthogonal Jacobi Polynomial.

3. Approximate Solutions of Modified Generalized Hulthen Potential

The modified generalized Hulthen potential (MGHP) proposed in this work is defined as

$$V(r) = -V_0 + V_1 \left(\frac{a + b e^{-\alpha r}}{g + d e^{-\alpha r}} \right), \tag{5}$$

where V_0, V_1 are the strength of the potential, a, b, d, g are adjustable potential parameters and α is the screening parameter, the Hulthen potential is a short-range potential in physics which behaves like a coulomb potential for a small values of r and decreases exponentially for a larger values of r [22, 23]. This potential is very important in atom and molecular fields etc. [24-26].

The radial Klein-Gordon equation for a special case of equal scalar and vector potential is given as [28]

$$\left[E^2 - M^2(r) + 2(E + M(r))V(r) - \frac{\lambda}{r^2} \right] R(r) = 0, \tag{6}$$

where M is the mass, E is the relativistic energy, $V(r)$ is the potential under investigation and $\lambda = l(l+1)$, which is the separation constant. In this paper, we assume that the mass of the Klein-Gordon particle depends on the spatial coordinate as [29]

$$M(r) = M_0 + M_1 V(r) =$$

$$M_0 + M_1 \left[-V_0 + V_1 \left(\frac{a + b e^{-\alpha r}}{g + d e^{-\alpha r}} \right) \right] \tag{7}$$

The most extensive use of such kind of mass is in the physics of semiconductor quantum well structures [30]. If one set $M_0 = 0, a = 0, V_0 = 0, V_1 = b = g = 1$ and mapping $d \to b$, then our proposed mass is in agreement with the position dependent mass of Ref. [31]. The behavior of the mass function of Eq.(7) with position r is presented in Fig.1.

To obtain the eigenvalues and corresponding eigen functions for this system, we substitute Eqs. (5) and (7) into Eq. (6) to have:

$$\left[\begin{array}{c} \left[E^2 - \left(M_0 + M_1 \left[-V_0 + V_1 \left(\frac{a + b e^{-\alpha r}}{g + d e^{-\alpha r}} \right) \right] \right) \right] + \\ 2 \left(E + M_0 + M_1 \left[-V_0 + V_1 \left(\frac{a + b e^{-\alpha r}}{g + d e^{-\alpha r}} \right) \right] \right) \\ \left(-V_0 + V_1 \left(\frac{a + b e^{-\alpha r}}{g + d e^{-\alpha r}} \right) \right) - \frac{l(l+1)}{r^2} \end{array} \right] R(r) = 0. \tag{8}$$

Equation (8) has no exact solution for $l \neq 0$ due to the potential barrier, but can be solved approximately by using a suitable approximation scheme. Here we make use of an approximation scheme to deal with the potential barrier as [32]

$$\frac{1}{r^2} \approx \alpha^2 \left(\frac{g}{g + d e^{-\alpha r}} \right)^2, \tag{9}$$
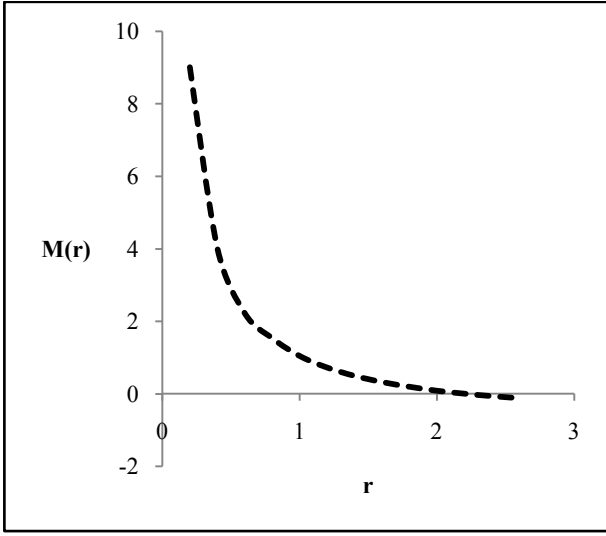
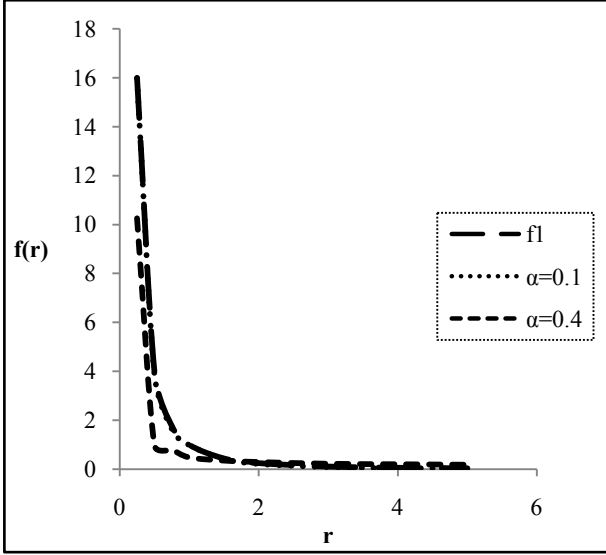**Fig. 1 Variation of mass function $M(r)$ with r.**

Fig. 2 Comparison of the potential barrier $f1 = \dfrac{1}{r^2}$ with the approximation for $\alpha = 0.1$ and 0.4

where $g = -d$. The comparison of the approximation scheme of Eq. (9) with the centrifugal term (potential barrier) for various values of α is discussed in Fig. 2. From the graph it is obvious that the approximation is suitable for short range potentials.

Substituting Eq. (9) into Eq. (8) and using the transformation $s = e^{-\alpha r}$ with a simple algebra we obtain the following equation:

$$\frac{d^2 R(s)}{ds^2} + \frac{\left(1 + \dfrac{d}{g} s\right)}{s\left(1 + \dfrac{d}{g} s\right)} \frac{dR}{dS} +$$

$$\frac{1}{s^2\left(1 + \dfrac{d}{g} s\right)^2}\left[-Q_1 s^2 + Q_2 s - Q_3\right] R(s) = 0, \tag{10}$$

where

$$-\varepsilon^2 = \frac{E_{nl}^2 - M_0^2}{\alpha^2} \tag{11}$$

$$Q_1 = \frac{d^2}{g^2}\varepsilon^2 + \frac{2(E + M_0)}{\alpha^2}\left(\frac{d^2}{g^2}V_0 - \frac{d}{c^2}V_1 b\right)$$
$$+ \frac{d^2}{g^2}\frac{M_1 V_0}{\alpha^2}(M_1 V_0 - 2M_0 - V_0)$$
$$+ \frac{d}{g^2}\frac{2M_1 V_1 b}{\alpha^2}(M_0 - M_1 V_0 + 2V_0) \tag{12}$$
$$- \frac{M_1 V_1^2}{\alpha^2}(2 - M_1)b^2$$

$$Q_2 = -\frac{2d}{g}\varepsilon^2 + \frac{2(E + M_0)}{\alpha^2}\left(\frac{d}{g^2}V_1 a - \frac{2V_0 d}{g} + \frac{V_1 b}{g}\right)$$
$$+ \frac{2d}{g}\frac{M_1 V_0}{\alpha^2}(2M_0 - M_1 V_0 + 2V_0)$$
$$+ \frac{d}{g^2}\frac{2M_1}{\alpha^2}(M_1 V_0 V_1 - M_0 - 2V_0 V_1)a \tag{13}$$
$$+ \frac{2}{g}\frac{M_1 V_1}{\alpha^2}(M_1 V_0 - M_0 - 2V_0)b$$
$$+ \frac{2M_1 V_1^2}{\alpha^2}(2 - M_1 a)b$$

$$Q_3 = \varepsilon^2 + \frac{2(E + M_0)}{\alpha^2}\left(V_0 - \frac{V_1}{g}a\right)$$
$$+ \frac{2}{g}\frac{M_1}{\alpha^2}(M_0 - M_1 V_0 V_1 - 2V_0 V_1)a$$
$$+ \frac{M_1 V_1^2}{\alpha^2}(M_1 - 2)a^2 - \tag{14}$$
$$\frac{M_1 V_0}{\alpha^2}(2V_0 - 2M_0 - M_1 V_0) + l(l + 1)$$

Comparing Eq. (10) with Eq. (2) and making use of

Eq. (5) the following parameters

$$c_1 = 1, \quad c_2 = c_3 = -\frac{d}{g},$$

$$\xi_1 = Q_1, \quad \xi_2 = Q_2, \quad \xi_3 = Q_3, \quad c_4 = 0, c_5 = \frac{d}{2g},$$

$$c_6 = \frac{d^2}{4g^2} + \frac{d^2\varepsilon^2}{g^2} + \frac{2(E+M_0)}{\alpha^2}\left(\frac{d^2}{g^2}V_0 - \frac{d}{g^2}V_1 b\right) + \frac{d^2}{g^2}\frac{M_1 V_0}{\alpha^2}(M_1 V_0 - 2M_0 - 2V_0)$$

$$+ \frac{d}{g^2}\frac{2M_1 V_1}{\alpha^2}(M_0 - M_1 V_0 + 2V_0)b - \frac{M_1 V_1^2}{\alpha^2}(2 - M_1)b^2$$

$$c_7 = \frac{2d}{g}\varepsilon^2 - \frac{2(E+M_0)}{\alpha^2}\left(\frac{d}{g^2}V_1 a - \frac{2V_0 d}{g} + \frac{V_1 b}{g}\right) - \frac{2d}{g}\frac{M_1 V_0}{\alpha^2}(2M_0 - M_1 V_0 + 2V_0)$$

$$- \frac{d}{g^2}\frac{2M_1}{\alpha^2}(M_1 V_0 V_1 - M_0 - 2V_0 V_1)a - \frac{2}{g}\frac{M_1 V_1}{\alpha^2}(M_1 V_0 - M_0 - 2V_0)b - \frac{2M_1 V_1^2}{\alpha^2}(2 - M_1 a)b$$

$$c_8 = \varepsilon^2 - 2\frac{(E+M_0)}{\alpha^2}\left(V_0 + \frac{V_1 a}{g}\right) + \frac{2}{g}\frac{M_1}{\alpha^2}(M_0 M_1 V_0 V_1 + 2V_0 V_1)a$$

$$+ \frac{M_1 V_1^2}{\alpha^2}(M_1 - 2)a^2 - \frac{M_1 V_0}{\alpha^2}(2V_0 + 2M_0 - M_1 V_0) + l(l+1)$$

$$c_9 = -4\frac{d^2}{g^2}\frac{(E+M_0)}{\alpha^2}v_0 + \frac{d}{g}2\frac{M_1 V_1^2}{\alpha^2}(2 - M_1 a)b - \frac{d^2}{g^2}\frac{M_1}{\alpha^2}(M_0 - M_1 V_0 V_1 + 2V_0 V_1)\frac{a}{g}$$

$$+ \frac{d^2}{g^2}\frac{M_1 V_1^2}{\alpha^2}(M_1 - 2)a^2 + \frac{d}{g^2}l(l+1) + \frac{d^2}{4g^2} - \frac{M_1 V_1^2}{\alpha^2}(2 - M_1)b^2$$

$$c_{10} = 1 + 2\left[\begin{array}{c} \varepsilon^2 - 2\dfrac{(E+M_0)}{\alpha^2}\left(V_0 + V_1\dfrac{a}{g}\right) + \\[2mm] \dfrac{2}{g}\dfrac{M_1}{\alpha^2}(M_0 - M_1 V_0 V_1 + 2V_0 V_1)a + \dfrac{M_1 V_1^2}{\alpha^2}(M_1 - 2)a^2 \\[2mm] -\dfrac{M_1 V_0}{\alpha^2}(2V_0 + 2M_0 - M_1 V_0) + l(l+1) \end{array}\right]^{\frac{1}{2}} \qquad c_{11} = -2\frac{d}{g} + 2\left(\sqrt{c_9} - \frac{d}{g}\sqrt{c_8}\right)$$

$$c_{12} = \left[\begin{array}{c} \varepsilon^2 - 2\dfrac{(E+M_0)}{\alpha^2}\left(V_0 + \dfrac{V_1}{g}a\right) + \dfrac{2}{g}\dfrac{M_1}{\alpha^2}(M_0 - M_1 V_0 V_1 + 2V_0 V_1)a + \dfrac{M_1 V_1^2}{\alpha^2}(M_1 - 2)a^2 \\[2mm] -\dfrac{M_1 V_0}{\alpha^2}(2V_0 + 2M_0 - M_1 V_0) + l(l+1) \end{array}\right]^{\frac{1}{2}}$$

$$c_{13} = \frac{d}{2g} - \left(\sqrt{c_9} - \frac{d}{g}\sqrt{c_8}\right) \qquad (15)$$

Substituting Eqs. (11) – (15) into Eq. (2), the energy eigenvalues for this system is obtained explicitly as

$$E_{nl}^2 - M_0^2 = \alpha^2 \delta - \frac{\alpha^2}{4} \left(\frac{\Lambda + 2\left(n + \frac{1}{2}\right)\beta}{\frac{d}{g}\left(n + \frac{1}{2}\right) - \beta} \right)^2, \tag{16}$$

where

$$\delta = -2\left(\frac{E + M_0}{\alpha^2} \right)\left(V_0 + \frac{V_1 a}{g} \right) + \frac{2}{\alpha^2 g} M_1 \left(M_0 - M_1 V_0 V_1 + 2V_0 V_1 \right) a$$
$$+ \frac{M_1 V_1^2}{\alpha^2}(M_1 - 2)a^2 - \frac{M_1 V_0}{\alpha^2}(2V_0 + 2M_0 - M_1 V_0) + l(l+1). \tag{17}$$

$$\beta = \left[\begin{array}{l} -4\frac{d^2}{g^2}\frac{(E+M_0)}{\alpha^2}v_0 + \frac{d}{g}2\frac{M_1 V_1^2}{\alpha^2}(2 - M_1 a)b - \frac{d^2}{g^2}\frac{M_1}{\alpha^2}(M_0 - M_1 V_0 V_1 + 2V_0 V_1)\frac{a}{g} \\[2mm] + \frac{d^2}{g^2}\frac{M_1 V_1^2}{\alpha^2}(M_1 - 2)a^2 + \frac{d^2}{g^2}l(l+1) + \frac{d^2}{4g^2} - \frac{M_1 V_1^2}{\alpha^2}(2 - M_1)b^2 \end{array} \right]^{\frac{1}{2}} \tag{18}$$

$$\Lambda = -\frac{d}{2g} - \frac{d}{g}n(n+1) + \frac{2d}{\alpha^2 g^2}\left((E + M_0)V_1 - M_1 M_0 - 2M_1 V_0 V_1 + M_1^2 V_0 V_1 \right)a$$
$$+ \frac{2}{\alpha^2 g}\left(2M_1 V_0 V_1 - 2g M_1 V_1^2 - (E + M_0)V_1 + M_1 M_0 V_1 - M_1^2 V_0 V_1 \right)b \tag{19}$$
$$+ \frac{2M_1^2 V_1^2}{\alpha^2}ab + \frac{2d}{\alpha^2 g}M_1 V_1^2(2 - M_1)a^2 + \frac{8d}{\alpha^2 g}(E + M_0)V_0 - \frac{2d}{g}l(l+1).$$

And the wave function is obtained using Eqs. (3) and (15) as

$$\Psi(r) = N_{nl}\left(e^{-\alpha r}\right)^{\mu}\left(1 + \frac{d}{g}e^{-\alpha r}\right)^{\frac{1}{2}+v}P_n^{(2\mu,2v)}\left(1 + 2\frac{d}{g}e^{-\alpha r}\right), \tag{20}$$

where

$$\mu = \left[\begin{array}{l} \varepsilon^2 - 2\frac{(E+M_0)}{\alpha^2}\left(V_0 + \frac{V_1 a}{g}\right) + \frac{2}{g}\frac{M_1}{\alpha^2}(M_0 M_1 V_0 V_1 + 2V_0 V_1)a \\[2mm] + \frac{M_1 V_1^2}{\alpha^2}(M_1 - 2)a^2 - \frac{M_1 V_0}{\alpha^2}(2V_0 + 2M_0 - M_1 V_0) + l(l+1) \end{array} \right]^{\frac{1}{2}},$$

$$v = -\frac{g}{d}\left[\begin{array}{l} -4\frac{d^2}{g^2}\frac{(E+M_0)}{\alpha^2}v_0 + \frac{d}{g}2\frac{M_1 V_1^2}{\alpha^2}(2 - M_1 a)b - \frac{d^2}{g^2}\frac{M_1}{\alpha^2}(M_0 - M_1 V_0 V_1 + 2V_0 V_1)\frac{a}{g} \\[2mm] + \frac{d^2}{g^2}\frac{M_1 V_1^2}{\alpha^2}(M_1 - 2)a^2 + \frac{d}{g^2}l(l+1) + \frac{d^2}{4g^2} - \frac{M_1 V_1^2}{\alpha^2}(2 - M_1)b^2 \end{array} \right]^{\frac{1}{2}}$$

and N_u is a normalization constant.

4. Results and Discussion

By setting $V_1 = 0,$ the potential in Eq. (5) reduces to constant potential [24]. If one set $V_0 = 0, a = g = 1, \ b = d = -1$ and using $\alpha - 2\alpha$ the potential under investigation reduces to Hulthen potential [10, 28]. Also setting $V_0 = 0, a = b = 1,$ $g = 1, d = -1$ and mapping $V_1 \rightarrow -V_1$ and $\alpha \rightarrow 2\alpha$ the potential in Eq.(5) becomes Rosen-Morse potential [33]. Woods-Saxon potential [34] could be deduced from our potential in Eq. (5) if one set $V_0 = a = 0, b = g = d = 1,$ map $V_1 \rightarrow -V_0$ and mapping $\alpha \rightarrow 2\alpha$. Similarly, if we set $V_0 = a = d = 0, b = g = 1$ and mapping $V_1 \rightarrow V_0$, our potential model reduces to Morse potential [35]. The energy spectrum for these potentials deduced from our potential models could be obtained by using the adjusted parameters in Eqs. (16, 17, 18 and 19).

5. Conclusion

In this paper, we have obtained appropriately the bound state solutions of the Klein-Gordon equation under equal and scalar modified generalized Hulthen potential with proper approximation to the centrifugal term (potential barrier) using a powerful NU technique. Explicitly, the energy eigenvalues and the corresponding wave functions expressed in terms of Jacobi polynomial are also obtained. Our approach here offers one of the few examples were the Klein-Gordon equation is solved approximately with position-dependent mass and in an external potential. Finally, in addition to the fundamental importance in physics, the solutions obtained here may play a very vital role in the study of hadrons for both theoretical and experimental physicists.

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