Journal of Mathematics and System Science 4 (2014) 34-36

A Short Note on the Conjugation in the Leibniz-Hopf Algebra

Neşet Deniz Turgay

Independent Scholar, 126/7 Sok. No:10/1, Daire 3, Evka 3, Bornova- Izmir 35050, Turkey

Received: Octorber 27, 2013 / Accepted: December 12, 2013 / Published: January 25, 2014.

Abstract: The Leibniz-Hopf algebra is the free associative Z – algebra with one generator in each positive degree and coproduct is given by the Cartan formula. It has been also known as the 'ring of noncommutative symmetric functions' [1], and to be isomorphic to the Solomon Descent algebra [12]. This Hopf algebra has links with algebra,topology and combinatorics. In this article we consider another approach of proof for the antipode formula in the Leibniz-Hopf algebra by using some properties of words in [2].

Key words: Hopf algebra, Leibniz-Hopf algebra, antipode, Steenrod algebra.

1. Introduction

The Leibniz-Hopf algebra, F, is the free associative $Z-$ algebra on one generator, S^n, $n>0$ in each degree with the graded Hopf algebra structure determined by $\Delta(S^n)=\sum_{i+j=n} S^i \otimes S^j$, $S^0=1$.

As F is a graded and connected algebra, we have: $F=\oplus_{n\geq 0} F_n$. Topologists know this algebra as the homology of the loop space of the suspension of the infinite complex projective space, $H_*\left(\Omega\Sigma CP^\infty\right)$. On the other hand, the antipode in $H_*\left(\Omega\Sigma CP^\infty\right)$ arises from the time-inversion of loops. Hence it leads to a geometric point of view for the antipode or conjugation, χ, on the Leibniz-Hopf algebra (see Section 1 of [3] for more details). The mod p reductions $F\otimes Z/p$ are also important in algebraic topology, since the mod p Steenrod algebra is naturally defined as a quotient of $F\otimes Z/p$ by the Adem Relations [4]. (to be precise, for odd primes, this is the sub algebra of the mod p Steenrod algebra generated by Steenrod powers.)

The conjugation on F and its dual has been studied in [5-7] connection with the Steenrod algebra and commutativity in ring spectra [8, 9]. A formula for the antipode on F was given in [10-12].

In this paper we give a proof for this conjugation formula in the language of words in [2]. This paper is produced from [13].

Corresponding author: Neşet Deniz Turgay, Ph.D., researcher, research fields: Algebraic topology, word Hopf algebras, Steenrod algebra. E-mail: deniz_turgay@yahoo.com.

2. Results

Proposition 2.1 Let R be the set of all words of degree n, where $n\geq 1$. Then

$$R=A_1\coprod A_2\coprod\cdots\coprod A_n,$$

where $A_i=\left\{\begin{matrix} i,l_1,\ldots,l_s : l_1,\ldots,l_s \text{ is a word of} \\ \text{degree } n-i \end{matrix}\right\}$ for $i=1,\ldots,n$.

Proof: See Proof of Proposition 2.3 of [2].

Lemma 2.2 The antipode for F may be defined by recursively $\chi(S^0)=S^0$, and for any $x\in F_n$, $n\geq 1$.

$$\chi(x)=-\sum_{i=1}^m y_i\chi(z_i),$$

where

$$\Delta(x)=S^0\otimes x+\sum_{i=1}^m y_i\otimes z_i$$

and degree of $z_i<n$.

Proof: See Proof of Lemma 2.4.12 of [13].

Proposition 2.3 For a given generator $x=S^n$, the antipode is given by

$$\chi(S^n) = \sum (-1)^k S^{i_1,\ldots,i_k},$$

where the summation is over all refinements [2] $S^{i_1,\ldots,i_k}$ of S^n.

Proof: The proof will proceed by induction on the degree *n*. Conjugation preserves the indentity element so we have: $\chi(S^0) = S^0$ [14].

On the other hand, by coproduct formula of F, Lemma 2.2 shows that we have a recursive formula for the antipode which is given by

$$\chi(S^n) = -\sum_{i=1}^{n} S^i \otimes \chi(S^{n-i}). \tag{1}$$

Eq. (1) expands in the following:

$$\chi(S^n) = -\left(S^1\chi(S^{n-1}) + \cdots + S^n\chi(S^{n-n})\right). \tag{2}$$

As $\chi(S^{n-n}) = S^0$, Eq. (2) turns into:

$$\chi(S^n) = -\begin{pmatrix} S^1\sum(-1)^{k_1} S^{r_{11},r_{12},\ldots,r_{1k_1}} + \\ +\cdots+S^n\sum(-1)^{k_n} S^{r_{n1},r_{n2},\ldots,r_{nk_n}} \end{pmatrix}, \tag{3}$$

where the summation is over all refinements $r_{11}, r_{12,}\ldots, r_{1k_1}$ of $n-1,\ldots,$ and the last one is over all refinements $r_{n1}, r_{n2,}\ldots, r_{nk_n}$ of $n-n=0$, i.e., 'empty' word. Hence, the length of $r_{n1}, r_{n2,}\ldots, r_{nk_n}$, namely k_n is zero. In particular, by distributive property of the product on F Eq. (3) turns into:

$$\chi(S^n) = \sum(-1)^{k_1+1} S^{1,r_{11},r_{12},\ldots,r_{1k_1}} + (-1)^{k_2+1} S^{1,r_{21},r_{22},\ldots,r_{2k_2}} \cdots + (-1)^{k_n+1} S^n. \tag{4}$$

In the language of Proposition 2.1, we observe that each summation on the right-hand side of Eq. (4) is over A_i, where $i = 1,\ldots,n$ and each summand in the summation appears with coefficients $(-1)^{k_i+1}$, where k_i+1 is the length of summand. I.e., the summation $\sum(-1)^{k_1+1} S^{1,r_{11},r_{12},\ldots,r_{1k_1}}$ is over A_1, and each summand $S^{1,r_{11},r_{12},\ldots,r_{1k_1}}$ has coefficient $(-1)^{k_1+1}$, and so on. Note that when $i=n$ the summation has only one summand which is S^n with coefficient $(-1)^{k_n+1} = (-1)^1$, since $k_n = 0$.

Moreover, the set of all refinements of the word n corresponds to R which is the finite union of these A_i. Thus, by Proposition 2.1 the right hand-side of Eq. (4) is the sum of all refinements of S^n.

Theorem 2.4 Let $S^{b_1,b_2,\ldots,b_p} \in F$, then the antipode is given by

$$\chi(S^{b_1,\ldots,b_p}) = \sum(-1)^n S^{t_1,\ldots,t_n},$$

where the summation is over all refinements $S^{t_1,\ldots,t_n}$ of $S^{b_p,\ldots,b_1}$.

Proof: As χ is also an anti-automorphism, we arrive at

$$\chi(S^{b_1,\ldots,b_p}) = \chi(S^{b_p})\chi(S^{b_{p-1}})\cdots\chi(S^{b_2})\chi(S^{b_1}).$$

More precisely, by Proposition 2.3 we have:

$$\chi(S^{b_1,\ldots,b_p}) = \sum(-1)^{k_1} S^{i_1,\ldots,i_{k_1}} \sum(-1)^{(k_2-k_1)} S^{i_{k_1+1},\ldots,i_{k_2}} \cdots \sum(-1)^{(k_p-k_{p-1})} S^{i_{k_{p-1}+1},\ldots,i_{k_p}}, \tag{5}$$

where $S^{i_1,\ldots,k_1}$ is a refinement of S^{b_p}, similarly, $S^{i_{k_1+1},\ldots,i_{k_2}}$ is a refinement of $S^{b_{p-1}},\cdots,S^{i_{k_{p-1}+1},\ldots,i_{k_p}}$, is a refinement of S^{b_1}. Multiplication of F is concatenation, so Eq. (5) turns into:

$$\chi(S^{b_1,\ldots,b_p}) = \sum(-1)^{k_p} S^{i_1,\ldots,i_{k_1},i_{k_1+1},\ldots,i_{k_2},i_{k_2+1},\ldots,i_{k_{p-1}},i_{k_{p-1}+1},\ldots,i_{k_p}},$$

where $S^{i_1,\ldots,i_{k_1},i_{k_1+1},\ldots,i_{k_2},i_{k_2+1},\ldots,i_{k_{p-1}},i_{k_{p-1}+1},\ldots,i_{k_p}}$ is a refinement of $S^{b_p,\ldots,b_1}$. This completes the proof.

Acknowledgments

I would like to thank M.D. Crossley for his assistance during my work on my Ph.D. degree.

References

[1] M. Hazewinkel, The algebra of quasi-symmetric functions is free over the integers, Adv. Math. 164 (2001) 283–300.

[2] N.D.Turgay, Some notes on the dual Leibniz-Hopf algebra, Journal of Mathematics and System Science 3 (2013) 573-576.

[3] A. Baker, B. Richter, Quasisymmetric functions from a topological point of view, Math. Scand. 103 (2008) 208–242.

[4] N.E. Steenrod, Cohomology Operations. Ann. Math. Studies, Vol. 50. Princeton: Princeton Univ. Press.1962.

[5] M.D. Crossley, N.D. Turgay, Conjugation Invariants in the mod 2 dual Leibniz-Hopf algebra, Communications in Algebra 41 (2013) 3261-3266.

[6] M.D. Crossley, N.D. Turgay, Conjugation Invariants in the Leibniz Hopf Algebra Journal of Pure and Applied Algebra 217 (2013) 2247-2254.

[7] N.D. Turgay, A note on conjugation invariants in the dual Leibniz-Hopf algebra 7, no. 17, 809-815 DOI: 10.12988/ija.2013.31099

[8] M.D. Crossley, S. Whitehouse, On conjugation invariants in the dual Steenrod algebra, P Am Math Soc, 128 (2000) 2809-2818.

[9] M.D. Crossley, S. Whitehouse, Higher conjugation in commutative Hopf algebras, P Edinburgh Math Soc, 44 (2) (2001) 19-26.

[10] M.D. Crossley, Some Hopf Algerabs of Words, Glasg. Math. J. 48 (2006) 575-582.

[11] R. Ehrenborg, On posets and Hopf algebras, Adv. Math. 119 (1996) 1-25.

[12] C. Malvenuto, C. Reutenauer, Duality between Quasi-Symmetric Functions and the Solomon Descent Algebra, J. Algebra 177 (1995) 967-982.

[13] N.D. Turgay, Conjugation invariants in Word Hopf algebras, Ph.D. Thesis, Swansea University, Swansea, 2012.

[14] M.E. Sweedler Hopf algebras, Benjamin, 1969.