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Differential Groupoids

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Abstract: The basic properties and some examples of the differential groupoids are studied.

Key words: Differential space, groupoid.

1. Introduction

In 1981 the notion of *differential group* and the notion of *group differential structure* (based on the notion of Sikorski's differential space – see [9]) was introduced and investigated by the second author in his PhD thesis [4]. Independently, in the same time, an analogous notions was investigated by P. Multarzyński in his PhD thesis (prepared in the Jagiellonian University in Krakov). Some results of this works have been published in [5], [6], [7], and [3] however most of them have never been presented in in the form of an article. Meanwhile, during last ten years, an interest in the theory of differential groups and groupoids appeared, concerned in applications of them in general relativity and cosmology (see references in [8]). This article is the first of the series of papers concerning differential groupoids and describing main results and many details of the theory of differential groups.

Section 2 of the paper contains basic definitions concerning theory of groupoids and theory of differential spaces. Basic definition and facts concerning groupoids can be find in [10] and [11] whereas foundations of theory of differential spaces can be find in [9]. In Section 3 we give the definition of a differential groupoid which is illustrated by an elementary example. Section 4 contains two another examples of topological and differential groupoids.

Without any other explanation we use the following symbols: $\mathbb{N}$-the set of natural numbers; $\mathbb{Z}$-the set of integers; $\mathbb{R}$-the set of reals.

2. Preliminaries

Definition 1. The sequence $(G, X, \alpha, \beta, m, \varepsilon, \tau)$ is called a *groupoid* G *over the base* X if G and X are arbitrary nonempty sets and: (i) the map $\alpha: G \to X$ called a *target* and the map $\beta: G \to X$ called a *source* are surjections; (ii) the map $m: G^{(2)} \to G$, where

$$G^{(2)} := \{(g,h) \in G \times G: \beta(g) = \alpha(h)\},$$

called a *multiplication* satisfies the following conditions:

- $(gh)k = g(hk)$ - *associativity*,
- $\alpha(gh) = \alpha(g)$ and $\beta(gh) = \beta(h)$

for each $g, h, k \in G$ (instead of $m(g,h)$ we write gh); (iii) the embedding $\varepsilon: X \to G$ called *the identity section* is such that:

$$\varepsilon(\alpha(g))g = g = g\varepsilon(\beta(g)),$$

$$\alpha(\varepsilon(x)) = x = \beta(\varepsilon(x))$$

for each $g \in G$ and $x \in X$;

(iv) the map $\tau: G \to G$ (denote by $g^{-1} = \tau(g)$) called the *inverce map* is such that

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$$g\tau(g) = \varepsilon(\alpha(g)) \text{ and } \tau(g)g = \varepsilon(\beta(g))$$

for each $g \in G$.

For the definition, basic properties and applications of groupoids see [10] or [11].

Definition 2. A *subgroupoid* of the groupoid $(G, X, \alpha, \beta, m, \varepsilon, \tau)$ is a sequence $(H, \alpha(H), \alpha_{|H}, \beta_{|H}, m_{|H^{(2)}}, \varepsilon_{\alpha(H)}, \tau_{|H})$, where H is nonempty subset of G which is closed under the multiplication and the inverse i. e. (i) if $g, h \in H$ and $(g, h) \in G^{(2)}$, then $gh \in H$; (ii) if $h \in H$, then $h^{-1} \in H$.

Definition 3. The groupoid $(G, X, \alpha, \beta, m, \varepsilon, \tau)$ over the set X is called a *topological groupoid*, if G and X are topological spaces, X is a Hausdorff space and the mappings $\alpha, \beta, m, \varepsilon$ and τ are continuous. Then τ is a homeomorphism.

We recall now the definition of a (Sikorski's) differential space. Let M be a nonempty set and let $\mathcal{C}$ be a family of real valued functions on M. Denote by $\tau_{\mathcal{C}}$ the weakest topology on M with respect to which all functions of $\mathcal{C}$ are continuous. A subbase of the topology $\tau_{\mathcal{C}}$ consists of sets of the form

$$\{p: f(p) < a\} \text{ and } \{p: f(p) > a\},$$

where $a \in \mathbb{R}$ and $f \in \mathcal{C}$. A function $f: M \to \mathbb{R}$ is called *a local $\mathcal{C}$-function on M* if for every $m \in M$ there is a neighborhood V of m and $\alpha \in \mathcal{C}$ such that $f_{|V} = \alpha_{|V}$. The set of all local $\mathcal{C}$-functions on M is denoted by $\mathcal{C}_M$. Note that any function $f \in \mathcal{C}_M$ is continuous with respect to the topology $\tau_{\mathcal{C}}$. Then $\tau_{\mathcal{C}_M} = \tau_{\mathcal{C}}$ (see [1], [2]).

A function $f: M \to \mathbb{R}$ is called *$\mathcal{C}$-smooth function on M* if there exist $n \in \mathbb{N}, \omega \in C^\infty(\mathbb{R}^n)$ and $\alpha_1, \dots, \alpha_n \in \mathcal{C}$ such that

$$f = \omega \circ (\alpha_1, \dots, \alpha_n).$$

The set of all $\mathcal{C}$-smooth functions on M is denoted by $sc\mathcal{C}$. Since $\mathcal{C} \subset sc\mathcal{C}$ and any superposition $\omega \circ (\alpha_1, \dots, \alpha_n)$ is continuous with respect to $\tau_{\mathcal{C}}$ we obtain $\tau_{sc\mathcal{C}} = \tau_{\mathcal{C}}$ (see [1], [2]).

Definition 4. A set $\mathcal{C}$ of real functions on M is said to be a (*Sikorski's*) *differential structure* if: (i) $\mathcal{C}$ is *closed with respect to localization* i.e. $\mathcal{C} = \mathcal{C}_M$; (ii) $\mathcal{C}$ is closed with respect to superposition with smooth functions i.e. $\mathcal{C} = sc\mathcal{C}$.

In this case a pair $(M, \mathcal{C})$ is said to be a *(Sikorski's) differential space* (see [9]). Any element of $\mathcal{C}$ is called a *smooth function on M* (with respect to $\mathcal{C}$).

It is easy to prove that the intersection of any family of differential structures defined on a set $M \neq \emptyset$ is a differential structure on M (see [1], [2], Proposition 2.1).

Let $\mathcal{F}$ be a set of real functions on M. Then the intersection $\mathcal{C}$ of all differential structures on M containing $\mathcal{F}$ is a differential structure on M. It is the smallest differential structure on M containing $\mathcal{F}$. One can easy prove that $\mathcal{C} = (\mathrm{sc}\mathcal{F})_{\mathrm{M}}$. This structure is called *the differential structure generated by* $\mathcal{F}$ and is denoted by $gen(\mathcal{F})$. Functions of $\mathcal{F}$ are called *generators* of the differential structure $\mathcal{C}$. We have also $\tau_{(sc\mathcal{F})_M} = \tau_{sc\mathcal{F}} = \tau_{\mathcal{F}}$.

Let $(M, \mathcal{C})$ and $(N, \mathcal{D})$ be differential spaces. A map $F: M \to N$ is said to be *smooth* if for any $\beta \in \mathcal{D}$ the superposition $\beta \circ F \in \mathcal{C}$. We will denote the fact that $\mathcal{F}$ is smooth writing

$$F: (M, \mathcal{C}) \to (N, \mathcal{D}).$$

If $F: (M, \mathcal{C}) \to (N, \mathcal{D})$ is a bijection and $F^{-1}: (N, \mathcal{D}) \to (M, \mathcal{C})$ then F is called a *diffeomorphism*.

If A is a nonempty subset of M and $\mathcal{C}$ is a differential structure on M then $\mathcal{C}_A$ denotes the differential structure on A generated by the family of restrictions $\{\alpha_{|A}: \alpha \in \mathcal{C}\}$. The differential space $(A, \mathcal{C}_A)$ is called *a differential subspace* of $(M, \mathcal{C})$. One can easy prove that if $(M, \mathcal{C})$ and $(N, \mathcal{D})$ are differential spaces and $F: M \to N$ then $F: (M, \mathcal{C}) \to (N, \mathcal{D})$ iff $F: (M, \mathcal{C}) \to (F(M), F(M)_{\mathcal{D}})$.

If the map $F: (M, \mathcal{C}) \to (F(M), F(M)_{\mathcal{D}})$ is a diffeomorphism then we say that $F: M \to N$ is *a diffeomorphism onto its range* (in $(N, \mathcal{D})$). In particular the natural embedding

$$A \ni m \mapsto i(m) := m \in M$$

is a diffeomorphism of $(A, \mathcal{C}_A)$ onto its range in $(M, \mathcal{C})$.

If $\{(M_i, \mathcal{C}_i)\}_{i\in I}$ is an arbitrary family of differential spaces then we consider the Cartesian product

$$\prod_{i\in I} M_i$$

as a differential space with the differential structure

$$\widehat{\bigotimes_{i\in I}} \mathcal{C}_i$$

generated by the family of functions

$$\mathcal{F} := \{\alpha_i \circ pr_i : i \in I, \alpha_i \in \mathcal{C}_i\},$$

where

$$\prod_{i\in I} M_i \ni (m_i) \mapsto pr_j\big((m_i)\big) =: m_j \in M_j$$

for any $j \in I$. The topology

$$\tau \widehat{\bigotimes_{i\in I}} \mathcal{C}_i$$

coincides with the standard product topology on

$$\prod_{i\in I} M_i.$$

We will denote the differential structure

$$\widehat{\bigotimes_{i\in I}} \mathcal{C}^\infty(\mathbb{R})$$

on $\mathbb{R}^I$ by $\mathcal{C}^\infty(\mathbb{R}^I)$. In the case when I is an n-element finite set the differential structure $\mathcal{C}^\infty(\mathbb{R}^I)$ coincides with the ordinary differential structure $\mathcal{C}^\infty(\mathbb{R}^n)$ of all real-valued functions on $\mathbb{R}^n$ which posses partial derivatives of any order (see [9]). In any case a function $\alpha: \mathbb{R}^I \to \mathbb{R}$ is an element of $\mathcal{C}^\infty(\mathbb{R}^I)$ iff for any $a = (a_i) \in \mathbb{R}^I$ there are $n \in \mathbb{N}$, elements $i_1, i_2, \cdots, i_n \in I$, a set U open in $\mathbb{R}^n$and a function $\omega \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that

$$a \in U[i_1, i_2, \cdots, i_n] := \{(x_i) \in \mathbb{R}^I : (x_{i_1}, x_{i_1}, \cdots, x_{i_n}) \in U\}$$

and for any $x = (x_i) \in U[i_1, i_2, \cdots, i_n]$ we have

$$\alpha(x) = \omega(x_{i_1}, x_{i_1}, \cdots, x_{i_n}).$$

Let $\mathcal{F}$ be a family of generators of a differential structure $\mathcal{C}$ on a set M. *The generator embedding* of the differential space $(M, \mathcal{C})$ into the Cartesian space defined by $\mathcal{F}$ is a mapping

$$\phi_{\mathcal{F}}: (M, \mathcal{C}) \to \left(\mathbb{R}^{\mathcal{F}}, C^\infty(\mathbb{R}^{\mathcal{F}})\right)$$

given by the formula

$$\phi_{\mathcal{F}}(m) = \big(\alpha(m)\big)_{\alpha\in\mathcal{F}}$$

(for example if $\mathcal{F} = \{\alpha_1, \alpha_2, \alpha_3\}$ then $\phi_{\mathcal{F}}(m) = \big(\alpha_1(m), \alpha_2(m), \alpha_3(m)\big) \in \mathbb{R}^3 \cong \mathbb{R}^{\mathcal{F}}$). If $\mathcal{F}$ separates points of M the generator embedding is a diffeomorphism onto its image. On that image we consider a differential structure of a subspace of $\left(\mathbb{R}^{\mathcal{F}}, C^\infty(\mathbb{R}^{\mathcal{F}})\right)$ (see [2], Proposition 2.3).

3. Basic Properties of Differential Groupoids

Definition 5. Let $(G, X, \alpha, \beta, m, \varepsilon, \tau)$ be a groupoid. A differential structure $\mathcal{C}$ on G is called *a groupoid differential structure*, if the following conditions are satisfied: (i) the multiplication map $m: G^{(2)} \to G$ is smooth with respect to the differential structure of the differential subspace on $G^{(2)} \subset G \times G$; (ii) the inverse map $\tau: G \to G$ and the mappings $\varepsilon \circ \alpha: G \to G$ and $\epsilon \circ \beta: G \to G$ are smooth..

A groupoid G equipped with a groupoid differential structure $\mathcal{C}$ is called *a differential groupoid*.

On $G \times G$ we consider natural differential structure of the Cartesian product which we denote by $\mathcal{C} \widehat{\otimes} \mathcal{C}$.

Example 1. Let $(X, \mathcal{D})$ be a differential space. Then the groupoid of pairs ($G = X \times X$ – see [11]) with the differential structure $\mathcal{C} := \mathcal{D} \widehat{\otimes} \mathcal{D}$ is a differential groupoid.

Let $\mathcal{C}$ be a groupoid differential structure on a groupoid $(G, X, \alpha, \beta, m, \varepsilon, \tau)$. We know that $\varepsilon(X) \subset G$. On the set $\varepsilon(X)$ there exists the structure of differential subspace of G, i. e. $\mathcal{C}_{\varepsilon(X)}$. Then we will consider X as a support of the differential space $(X,\mathcal{D})$, where the differential structure $\mathcal{D}=\{f\circ\varepsilon: f\in\mathcal{C}_{\varepsilon(X)}\}$ is said to be *induced from the differential structure* $\mathcal{C}_{\varepsilon(X)}$ (or $\mathcal{C}$) *by the map* ε. One can easy show that the identity section ε, the target map α and the source map β are smooth with respect to $\mathcal{D}$ i. e. $\varepsilon:(X,\mathcal{D})\to(G,\mathcal{C})$ and $\alpha,\beta:(G,\mathcal{C})\to(X,\mathcal{D})$.

Let H be a subgroupoid of a groupoid G endowed with a groupoid differential structure $\mathcal{C}$. It is easy to show that the set $\mathcal{C}_H$ is a groupoid differential structure on H. Then the pair $(H,\mathcal{C}_H)$ is called *a differential subgroupoid* of the differential groupoid $(G,\mathcal{C})$. We will write shortly then H is a differential subgroupoid of a differential groupoid G.

4. Examples of Topological and Differential Groupoids

Example 2. Let G be a set of all diffemorphisms between compact subsets of $\mathbb{R}^n$. For arbitrary element g of the set G we have: $g:(K_1,\mathcal{C}^\infty(\mathbb{R}^n)_{K_1})\to(K_2,\mathcal{C}^\infty(\mathbb{R}^n)_{K_2})$, where K_1 i K_2 are compact subsets in $\mathbb{R}^n$ or shortly $g:K_1\to K_2$. We denote by X_n the family of all non-empty compact subsets in $\mathbb{R}^n$. Let the value of the $\alpha:G\to X_n$ at the element $g\in G$ be the image of the map $g:K_1\to K_2$, i. e. $\alpha(g)=K_2$, and the value of the map $\beta:G\to X_n$ at g be the domain of the diffeomorphism g, i. e. $\beta(g)=K_1$. The value of the embedding $\varepsilon:X_n\to G$ for each compact set $K\in X_n$ is the identity map i. e. $\varepsilon(K)=id_K$. The value of the map $\tau:G\to G$ at $g\in G$ is equal to the inverse map i. e. $\tau(g)=g^{-1}$. As before we put $G^{(2)}=\{(g_1,g_2)\in G^2:\beta(g_1)=\alpha(g_2)\}$. The multiplication m: $G^{(2)}\to G$ is defined by equation: $m(g_1,g_2)=g_1\circ g_2$, where $\circ$ is an ordinary mappings composition. Then the sequence $(G,X_n,\alpha,\beta,m,\varepsilon,\tau)$ is a groupoid.

Let for any two compact sets $K_1,K_2\in X_n$ and any two diffeomorphisms $g_1,g_2\in G$

$$d_n(K_1,K_2)=\sup_{x\in K_1}\left(\inf_{y\in K_2}\|x-y\|\right)$$
$$=\sup_{y\in K_2}\left(\inf_{x\in K_1}\|x-y\|\right)$$

and

$$\tilde{d}_n(g_1,g_2)=d_{2n}(graf\ g_1,graf\ g_2).$$

Then d_n and $\tilde{d}_n$ are metrics on X_n and G, respectively. Mappings $\alpha,\beta,m,\varepsilon$ and τ are continuous with respect to the topology τ_{d_n} and $\tau_{\tilde{d}_n}$ given by metrics d_n and $\tilde{d}_n$ respectively. Hence $(G,X_n,\alpha,\beta,m,\varepsilon,\tau)$ is a topological groupoid. We will denote this groupoid by $DCS(\mathbb{R}^n)$.

Example 3. Let G be such as in Example 2 and let $G_0\subset G$ contains all diffeomorphisms $g\in G$ for which the domain D_g is the cloasure of its interior i.e.

$$D_g=cl\left(int(D_g)\right).$$

Let us consider the set

$$\tilde{G}=\{(g,a)\in G_0\times\mathbb{R}^n: a\in D_g\},$$

where $n\in\mathbb{N}$ is constant.

As the base of the groupoid $G=(\tilde{G},X,\alpha,\beta,m,\varepsilon,\tau)$ we take the set X composed of all pairs (K,a), where K is a compact subset of $\mathbb{R}^n$ and $a\in K$.

The source and target maps are defined in the following way:

$$\alpha(g,a)=\left(R_g,g(a)\right)\text{ oraz }\beta(g,a)=(D_g,a),$$

where D_g is the domain and R_g is the image of the diffeomorphism g.

Groupoid action m on pairs (g,a) and (h,b) is done, if $D_h=R_g$ and $b=g(a)$. Then $m((h,b),(g,a))=(h\circ g,a)$. The identity secion ε we define by:

- $\varepsilon(K,a)=(id_K,a)$ for any $(K,a)\in X$ and the inverse map τ has the value
- $\tau(g,a)=(g^{-1},g(a))$ for any $(g,a)\in\tilde{G}$. On the set $\tilde{G}$ we consider the family of
- functions $\mathcal{F}=\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3$, where $\mathcal{F}_1$ is a

family of functions of the form

- $f_\eta(g,a) = \eta(g(a))$ for $(g,a) \in \tilde{G}$ and $\eta \in C^\infty(\mathbb{R}^n)$,

$\mathcal{F}_2$ is a family of functions δ_i, where for any multiindex $i \in (\mathbb{N} \cup \{0\})^n$we have

$$\delta_i(g,a) = \frac{\partial^{|i|} g}{\partial x^i}(a) \text{ for } (g,a) \in \tilde{G}$$

(all partial derivatives exists because $D_g = cl\left(int(D_g)\right)$) and $\mathcal{F}_3$ is a family of functions p_η defined by

$$p_\eta\ (g,a) = \eta(a) \text{ for } (g,a) \in \tilde{G},$$

where $\eta \in C^\infty(\mathbb{R}^n)$.

The family of functions $\mathcal{F}$ generates the differential structure $\mathcal{C}$ on $\tilde{G}$ ($\mathcal{C} = gen\,\mathcal{F}$).

We will prove that $\mathcal{C}$ is a groupoid differential structure on $\tilde{G}$. For it is enough to show that each compositions of functions from families $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ with mappings $m, \tau, \varepsilon \circ \alpha$ and $\varepsilon \circ \beta$ are smooth.

Let us take an arbitrary function $f_\eta \in \mathcal{F}_1$. Then we have

$$f_\eta\left(m\big((h,b),(g,a)\big)\right) = f_\eta(h \circ g, a)$$

$$= \eta\big((h \circ g)(a)\big) = \eta\left(h\big(g(a)\big)\right)$$

$$= \eta\big(h(b)\big) = f_\eta(h,b)$$

which means that $\left(f_\eta \circ m\right)(\xi,\sigma) = f_\eta(\xi)$ for all $(\xi,\sigma) \in \tilde{G}^2$. Hence $f_\eta \circ m$ is an element of the differential structure $\mathcal{C}\widehat{\otimes}\mathcal{C}$ on the space $\tilde{G}^2$. We have also

$$f_\eta\big(\tau(g,a)\big) = f_\eta\big(g^{-1}, g(a)\big) = \eta\left(g^{-1}\big(g(a)\big)\right)$$

$$= \eta(a) = p_\eta(g,a)$$

for each element $(g,a) \in \tilde{G}$ which means that $f_\eta \circ \tau = p_\eta \in \mathcal{F}_3 \subset \mathcal{C}$.

Subsequently, for each element $(g,a) \in \tilde{G}$

$$f_\eta\big((\varepsilon \circ \alpha)(g,a)\big) = f_\eta\left(\varepsilon\big(\alpha(g,a)\big)\right) =$$

$$f_\eta\left(\varepsilon\left(R_g, g(a)\right)\right) = f_\eta\left(id_{R_g}, g(a)\right) =$$

$$\eta\left(id_{R_g}\big(g(a)\big)\right) = \eta\big(g(a)\big) = f_\eta(g,a).$$

Then $f_\eta \circ (\varepsilon \circ \alpha) = f_\eta \in \mathcal{F}_1 \subset \mathcal{C}$. Similarly, for each element $(g,a) \in \tilde{G}$

$$f_\eta\big((\varepsilon \circ \beta)(g,a)\big) = f_\eta\left(\varepsilon\big(\beta(g,a)\big)\right) = f_\eta\left(\varepsilon\big(D_g, a\big)\right)$$

$$= f_\eta\left(id_{D_g}, a\right) = \eta\left(id_{D_g}(a)\right)$$

$$= \eta(a) = p_\eta(g,a)$$

which means that

$$f_\eta \circ (\varepsilon \circ \beta) = p_\eta \in \mathcal{F}_3 \subset \mathcal{C}.$$

Now we can make similar considerations for an arbitrary function

$$\delta_i\left(m\big((h,b),(g,a)\big)\right) = \delta_i(h \circ g, a) =$$

$$= \frac{\partial^{|i|}}{\partial x^i}(h \circ g)(a) =$$

$$= \sum_{\substack{1 \le |j| \le |i| \\ |k_1| + \cdots + |k_n| = |j| \\ |s_1| + \cdots + |s_n| = |i|}} c^{k_1 \cdots k_n}_{j, s_1, \cdots, s_n} \frac{\partial^{|j|}}{\partial y^j}\Big(h\big(g(a)\big)\Big)\left(\frac{\partial^{|s_1|}}{\partial x^{s_1}} g(a)\right)^{k_1} \cdots \left(\frac{\partial^{|s_n|}}{\partial x^{s_n}} g(a)\right)^{k_n}$$

$$= \sum_{\substack{1\le|j|\le|i| \\ |k_1|+\cdots+|k_n|=|j| \\ |s_1|+\cdots+|s_n|=|i|}} c_{j,s_1,\cdots,s_n}^{k_1\cdots k_n}\,\delta_j(h,b)\delta_{s_1}^{k_1}(g,a)\cdots\delta_{s_n}^{k_n}(g,a)$$

where $c_{j,s_1,\cdots,s_n}^{k_1\cdots k_n} \in \mathbb{Z}$. Then $\delta_i \circ m$ is a polynomial function of several variables composed with elements of $\mathcal{F}_2$ and because of that it is an element of $\mathcal{C}$.

Let us consider the superposition $\delta_i \circ \tau$. We have

$$\delta_i\big(\tau(g,a)\big) = \delta_i\big(g^{-1}, g(a)\big) =$$

$$= \frac{\partial^{|i|}}{\partial x^i} g^{-1}\big(g(a)\big).$$

It is know from the course of calculus that the derivative $\frac{\partial^{|i|}}{\partial x^i} g^{-1}(b)$ is a rational function of partial derivatives of the map g taken at the point $g^{-1}(b)$. Then $\frac{\partial^{|i|}}{\partial x^i}(g^{-1})\big(g(a)\big)$ is a rational function of partial derivatives of the map g taken at the point a (which are elements of $\mathcal{F}_2$). Hence it belongs to $\mathcal{C}$.

Subsequently we consider the superposition $\delta_i((\varepsilon\circ\alpha))$.

$$\delta_i\big((\varepsilon\circ\alpha)(g,a)\big) = \delta_i\Big(\varepsilon\big(\alpha(g,a)\big)\Big)$$

$$= \delta_i\Big(\varepsilon\Big(R_g, g(a)\Big)\Big)$$

$$= \delta_i\Big(id_{R_g}, g(a)\Big) = \frac{\partial^{|i|}}{\partial x^i} id_{R_g}\big(g(a)\big)$$

$$= \frac{\partial^{|i|}}{\partial x^i} g(a) = \delta_i(g,a)$$

for each element $(g,a) \in \tilde{G}$. Then $\delta_i \circ (\varepsilon\circ\alpha) = \delta_i \in \mathcal{F}_2 \subset \mathcal{C}$. Similarly

$$\delta_i\big((\varepsilon\circ\beta)(g,a)\big) = \delta_i\Big(\varepsilon\big(\beta(g,a)\big)\Big) =$$

$$\delta_i\Big(\varepsilon\big(D_g, a\big)\Big) = \delta_i\Big(id_{D_g}, a\Big) = \frac{\partial^{|i|}}{\partial x^i} id_{D_g}(a) =$$

$constant$ (0 or 1).

Since $\delta_i((\varepsilon\circ\beta))$ is a constant function it belongs to $\mathcal{C}$.

Let's take any function $p_\eta \in \mathcal{F}_3$. Then we have

$$p_\eta\Big(m\big((h,b),(g,a)\big)\Big) = p_\eta(h\circ g, a) = \eta(a) =$$

$$p_\eta(g,a).$$

Then $(p_\eta \circ m)(\xi,\sigma) = p_\eta(\xi)$ for all $(\xi,\sigma) \in \tilde{G}^{(2)}$. It means that the superposition $p_\eta \circ m$ is a element of the differential structure $\mathcal{C}\widehat{\otimes}\mathcal{C}$ on the space $\tilde{G}^{(2)}$.

We have

$$p_\eta\big(\tau(g,a)\big) = p_\eta\big(g^{-1}, g(a)\big) = \eta\big(g(a)\big) = f_\eta(g,a)$$

for each element $(g,a) \in \tilde{G}$. Hence $p_\eta \circ \tau = f_\eta \in \mathcal{F}_1 \subset \mathcal{C}$. Since

$$p_\eta\big((\varepsilon\circ\alpha)(g,a)\big) = p_\eta\Big(\varepsilon\big(\alpha(g,a)\big)\Big)$$

$$= p_\eta\Big(\varepsilon\Big(R_g, g(a)\Big)\Big)$$

$$= p_\eta\Big(id_{R_g}, g(a)\Big) = \eta\big(g(a)\big)$$

$$= f_\eta(g,a)$$

for each element $(g,a) \in \tilde{G}$ we obtain that $p_\eta \circ (\varepsilon\circ\alpha) = f_\eta \in \mathcal{F}_1 \subset \mathcal{C}$. Similarly

$$p_\eta\big((\varepsilon\circ\beta)(g,a)\big) = p_\eta\Big(\varepsilon\big(\beta(g,a)\big)\Big)$$

$$= p_\eta\Big(\varepsilon\big(D_g, a\big)\Big) = p_\eta\Big(id_{D_g}, a\Big)$$

$$= \eta(a) = p_\eta(g,a)$$

for each element $(g,a) \in \tilde{G}$. It means that $p_\eta \circ (\varepsilon\circ\beta) = p_\eta \in \mathcal{F}_3 \subset \mathcal{C}$.

Finally we see that $\mathcal{C}$ is a groupoid differential structure on $\tilde{G}$ i. e. $(G,\mathcal{C})$ is a differential groupoid.

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